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Abstract

The type IIA and type IIB supergravity actions in $d = 10$ dimensions are the low-energy effective theories of type IIA and IIB string theory. In addition, the unique eleven dimensional supergravity theory is the low-energy effective action of M-theory. Higher order corrections to the low-energy effective actions of these supergravity theories contain perturbative and non-perturbative effects of the corresponding string theories and, as such, understanding the structure of the higher order terms provides an insight into the perturbative and non-perturbative formulations of string theory.

The U-duality groups of type IIA/B string theory and M-theory compactified on a torus to $d < 10$ dimensions puts powerful constraints on the higher derivative terms in the effective actions of these theories. In particular, the higher derivative terms in $d = 10 - n$ dimensions are required to possess coefficient functions that transform as $E_{n+1}(\mathbb{Z})$ automorphic forms. These automorphic forms are complex mathematical objects that encode all the perturbative and non-perturbative features of type II string theory and M-theory compactified on an torus to d dimensions.

We investigate the structure of the higher derivative terms and their associated automorphic forms in the effective actions of type IIA/B string theory and M-theory. Constraints on automorphic forms appearing in d dimensions by dimensional reduction of arbitrary higher derivative terms in the type IIA, type IIB and M-theory effective actions to d dimensions are obtained.

The behaviour of higher derivative terms in the d dimensional type II effective action in specific limits of various parameters is analysed. We derive a group theoretical interpretation for each limit. A general formula is given for a class of automorphic forms in these limits.

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1 Introduction

In the low-energy limit, the type II string theories and M-theory may be expressed as effective theories of their massless modes. The effective action S for type IIA/B string theory is an expansion in the Regge slope parameter α', which is related to the string tension T that acts as the coupling constant in the string world sheet action, by $T = \frac{1}{2\pi\alpha'}$. The effective action then has the schematic form

$$S = \alpha'^{-4}\left(S^{(0)} + \alpha' S^{(1)} + ... + \alpha'^{k} S^{(k)} + ...\right), \tag{1.1}$$

which, in the $\alpha' \to 0$ limit reduces to the corresponding two derivative type IIA [1–3] or type IIB [4–6] supergravity action. Analogously, the effective action of M-theory is an expansion in the eleven dimensional supergravity coupling κ_{11} that reduces to the two derivative eleven dimensional supergravity [7] action in the $\kappa_{11} \to 0$ limit. By dimensional analysis, the terms appearing in the effective action of type IIA/B string theory and M-theory at order k contain $2(k+1)$ derivatives.

The effective actions of type IIA/B string theory and M-theory are important as they encode all of the perturbative and non-perturbative effects of these theories. The two derivative terms that make up $S_{(0)}$ in equation (1.1) are fixed by supersymmetry and have been known for many years. However, only a handful of higher order terms in the type IIA/B string theory and M-theory effective actions, beyond the two derivative supergravity approximation, are known.

Several approaches have been adopted to determine the higher order terms in the effective actions of these theories. Firstly, string scattering amplitudes may be used to deduce the terms in the effective action that reproduces them. Another method is demanding conformal invariance on the string world sheet which requires all beta functions to vanish. The vanishing of the beta functions may then be interpreted as the equations of motion of the background fields and one may attempt to construct an action that reproduces these equations of motion. Thirdly, the effective actions of type IIA/B string theory and M-theory are strongly constrained by supersymmetry and, in principle, this may be used to deduce the higher order terms. Alternatively, one may use the duality symmetries of type IIA/B string theory and M-theory to investigate the structure of the higher order terms in the effective action.

When dimensionally reduced on a torus to $d < 10$ dimensions type IIA/B supergravity and eleven dimensional supergravity are equivalent. Moreover, it has been known for some time that type IIA/B supergravity compactified on an n torus, or, eleven dimensional supergravity compactified on an $n+1$ torus gives a $d = 10 - n$ dimensional maximal supergravity theory that possesses an $E_{n+1}(\mathbb{R})$ hidden symmetry [8–11]. The quantisation conditions on the brane charges [12, 13] ensures that only a discrete subgroup of these continuous symmetries may be preserved in the full theory. This led to the conjecture that the full type IIA/B string theory compactified on an n torus and M-theory compactified on an $n+1$ torus, to $d = 10 - n$ dimensions, is invariant under a restriction of $E_{n+1}(\mathbb{R})$ to an $E_{n+1}(\mathbb{Z})$ subgroup [14] known as the U-duality group. Demanding that the effective actions of type IIA/B string theory and M-theory are invariant under the $E_{n+1}(\mathbb{Z})$ U-duality group in $d = 10 - n$ dimensions is straightforward for the two derivative terms in the action. However, in the last 15 years it has been shown that the higher order terms require coefficient functions known as automorphic forms to guarantee this symmetry. The automorphic forms in the effective action of type IIA/B string theory and M-theory are now thought to control all the perturbative and non-perturbative features of these theories. This thesis is concerned with investigating the higher order terms and their associated automorphic forms in the effective action of type IIA/B string theory and M-theory.

The first conjecture involving automorphic forms in the effective action of type IIA/B string theory and M-theory was made in [15] where it was observed that a specific contraction of four Riemann curvatures, known as the R^4 term, that appears as a leading order correction to the effective action of type IIB string theory in ten dimensions picks up perturbative contributions at tree level and one loop from the scattering of four gravitons as well as non-perturbative D-instanton contributions. The metric used to construct the curvature R is invariant under $SL(2,\mathbb{Z})$ and the type IIB dilaton ϕ transforms non-trivially under $SL(2,\mathbb{Z})$. Therefore these perturbative contributions, which carry factors of $e^{-\frac{3}{2}\phi}$ and $e^{\frac{1}{2}\phi}$ in Einstein frame, at tree level and one loop respectively, explicitly break the expected $SL(2,\mathbb{Z})$ symmetry. However, the conjecture of [15] was that the perturbative and non-perturbative contributions combine to produce an $SL(2,\mathbb{Z})$ invariant coefficient function $\Phi_{SL(2)}$ for the R^4 term that takes the form

$$\Phi_{SL(2)} = \sum_{(m,n)\neq(0,0)} \frac{1}{\left((m - n\chi)^2 e^{\phi} + n^2 e^{-\phi}\right)^{\frac{3}{2}}}, \tag{1.2}$$

where χ is the type IIB axion. It should be noted that this function not only gives the correct powers of the string coupling $g_s = e^{\phi}$ for perturbative contributions at tree level and one loop and the non-perturbative contribution but also reproduces the exact numerical coefficients of these contributions. Moreover, the conjectured coefficient function implies that the type IIB R^4 term receives no perturbative contributions beyond one loop. Analysis of the four graviton scattering amplitude at two loops supports this observation [16–21]. Subsequent work has provided strong evidence that the conjecture in [15] is correct and that $\Phi_{SL(2)}$ is the unique type IIB R^4 coefficient function [22–24]. In particular it was shown via supersymmetry arguments that the automorphic form of the R^4 term in the type IIB effective action is constrained to satisfy a Laplace eigenvalue equation [24] and that the $s = \frac{3}{2}$ Eisenstein series in (1.2) is the unique solution to this equation [25]. The ability to extract this constraint on the R^4 term rests on the fact that the non-perturbative contribution to this term arises from $\frac{1}{2}$ BPS states. These supersymmetry arguments have been extended to the $\partial^4 R^4$ term [26] and the $\partial^6 R^4$ term [27] in the type IIB effective action, which receive non-perturbative contributions from $\frac{1}{4}$ BPS and $\frac{1}{8}$ BPS states, respectively.

The $\Phi_{SL(2)}$ coefficient function for the higher derivative R^4 term is the $SL(2,\mathbb{Z})$ Eisenstein series with $s = \frac{3}{2}$ and belongs to a larger class of functions known as automorphic forms. Further work has shown that the effective action of type IIA/B string theory and M-theory compactified to $d = 10 - n$ dimensions on an n torus contains terms with coefficient functions that are non-holomorphic $E_{n+1}(\mathbb{Z})$ automorphic forms [28–48], where the Dynkin diagram for the E_{n+1} Lie algebra is displayed in figure 1.

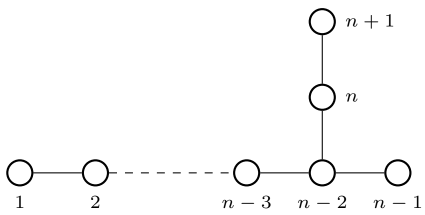

Figure 1: The E_{n+1} Dynkin diagram

The theory of automorphic forms is a complex and active research area. There are a variety of seemingly different ways to construct automorphic forms and it is not clear which constructions lead to automorphic forms that could appear in string theory or even whether different constructions of these automorphic forms are equivalent. A class of automorphic forms may be constructed by taking a representation of a group G along with a subgroup H [40]. For the effective action of type IIA/B string theory and M-theory compactified on an n torus the group G with subgroup H becomes the $E_{n+1}(\mathbb{R})$ group with a Cartan involution invariant subgroup H, where the scalars in the compactified theory parameterise an $E_{n+1}(\mathbb{R})/H$ coset. The representations one may take to construct an automorphic form that appears as a coefficient function in the effective action of type IIA/B string theory or M-theory in $d = 10 - n$ dimensions is an open problem. Furthermore, it is known that for the R^4 and $\partial^4 R^4$ terms, the construction of this class of non-holomorphic $E_{n+1}(\mathbb{Z})$ automorphic forms must be modified by certain constraints [37, 40], such that the resulting automorphic form satisfies Laplace eigenvalue equations on the $E_{n+1}(\mathbb{R})/H$ moduli space [45, 46] to appear as the coefficient function of these terms in $d < 7$ dimensions. This class of automorphic forms and the associated constraints are discussed in chapter 8.

Much of what is known so far about the higher order corrections to the effective action of type IIA/B string theory and M-theory is restricted to the lower order terms R^4, $\partial^4 R^4$ and $\partial^6 R^4$ that are protected by supersymmetry, although there are conjectures for individual higher order terms in the effective action of type IIB string theory in ten dimensions [33, 34]. In general, higher order terms in the effective action of type IIA/B string theory and M-theory beyond the R^4, $\partial^4 R^4$ and the $\partial^6 R^4$ are poorly understood. Critically, it is not known if terms of higher order than the $\partial^6 R^4$ term are protected by supersymmetry in the same way, or if some other unknown mechanism could induce similar non-renormalisation theorems.

Higher order terms in the effective action of type IIA/B string theory and M-theory in $d = 10-n$ dimensions are polynomials in the field strengths constructed from R-R or NS-NS gauge fields, fermionic fields, Cartan forms of the $E_{n+1}(\mathbb{R})/H$ moduli space, and contractions of Riemann curvatures with coefficient functions that transform as $E_{n+1}(\mathbb{Z})$ automorphic forms. Although outside of the work in [26, 34] little progress has been made in identifying the structure of these terms.

This thesis is based on the work contained in references [28–30] and is concerned with investigating various constraints placed upon the automorphic forms that appear as the coefficient functions of the higher derivative terms in the effective action of type IIA/B string theory and M-theory compactified on a torus to $d = 10 - n$ dimensions. The first several chapters reviews the concepts and techniques we use to study the higher derivative terms in the effective actions of type IIA/B string theory and M-theory. In the following chapters we implement the study of the higher derivative terms and their automorphic forms through these techniques.

In chapter two we review the relationship between the two derivative terms in the effective action of type IIA/B string theory and M-theory and the corresponding maximal supergravity theories in ten and eleven dimensions, emphasising the group theory constraining the maximal supergravity theories.

In chapter three we discuss the compactification process in the context of the effective action of type IIA/B string theory and M-theory along with the U and T dualities found upon dimensional reduction, in addition to the S duality of ten dimensional type IIB string theory.

Chapter four provides a brief review of the E_{11} formulation of supergravity which will be used to identify relationships between the fields in the different dimensionally reduced formulations of type IIA/B string theory and M-theory. In addition, the E_{11} formulation of supergravity is useful for deriving the correspondence between the limits of various parameters, such as the effective coupling string coupling g_d in d dimensions, and the decomposition of the E_{n+1} algebra of the U-duality group in terms of subalgebras preserved in these limits, this is explored in chapter eight. One should note that the conjectured E_{11} symmetry of type IIA/B and eleven dimensional supergravity is not required to hold for the results presented in this thesis to be valid.

In chapter five we take the techniques and results outlined in the first few chapters to dimensionally reduce the type IIB effective action to seven dimensions, where it is known that there exists an R^4 term with an $SL(5,\mathbb{Z})$ automorphic form that is a straightforward extension of the $SL(2,\mathbb{Z})$ Eisenstein series presented in equation (1.2). Note that in lower dimensions the construction of the automorphic form associated with the R^4 term becomes more complicated. We then seek to write the two derivative part of the dimensionally reduced action in $SL(5,\mathbb{Z})$ invariant form through expressing the $d = 7$ fields, that lie in representations of $SL(5,\mathbb{Z})$, in terms of $SL(5,\mathbb{Z})$ covariant building blocks constructed as non-linear representations of $SL(5,\mathbb{Z})$, namely the $SL(5)/SO(5)$ Cartan forms $\mathcal{S}$ and non-linearly realised field strengths $\mathcal{F}$ along with the $d = 7$, $SL(5,\mathbb{Z})$ invariant, curvature R. Using this construction we will put forward a conjecture on the relationship between automorphic forms of different higher derivative terms in the type IIB effective action in ten dimensions. It is also observed that the dimensional reduction of any higher order term in the effective action of type IIB string theory in ten dimensions leads to a higher derivative term in the $d = 7$ dimensional type IIB effective action, constructed from the same building blocks, that breaks the $SL(5, \mathbb{Z})$ symmetry.

The observation that the dimensional reduction of any higher order term in the effective action of type IIB string theory in ten dimensions to $d = 7$ leads to dimensionally reduced higher derivative terms that break the $SL(5, \mathbb{Z})$ symmetry motivated the work contained in reference [28] and presented in chapter six. In this chapter we demonstrate that the resolution to the breaking of the $SL(5, \mathbb{Z})$ symmetry by these dimensionally reduced higher derivative terms constrains the automorphic forms associated with the higher derivative terms of the effective action and applies for dimensional reduction to any dimension $d \geq 3$. In particular, restoring the $E_{n+1}(\mathbb{Z})$ symmetry in $d = 10 - n$ dimensions requires a higher order term in the d dimensional effective action to possess an automorphic form that contains a weight of $E_{n+1}(\mathbb{R})$. We also perform a similar calculation for M-theory and find that the higher order terms in the $d = 10 - n$ dimensional M-theory effective action possess an automorphic form that must contain a different weight of $E_{n+1}(\mathbb{R})$ from that found for the type IIB case.

Chapter seven is based on the work contained in reference [29]. In this chapter we use the E_{11} formulation of the maximal supergravity theories discussed in chapter four to carry out the same calculation as that of chapter six, for the type IIA theory, which is complicated by the fact that the type IIA dilaton mixes with the torus moduli upon dimensional reduction. However, the resulting constraint on the automorphic form of a higher derivative term in the $d = 10 - n$ dimensional type IIA effective action ends up being identical to that of the type IIB case. The equivalence of type IIA/B string theory and M-theory once compactified on a torus to $d < 10$ dimensions combined with the results of the calculations in each case provides conjectured constraints on automorphic forms of particular terms in the effective action.

In chapter eight the construction of a class of $E_{n+1}(\mathbb{Z})$ automorphic forms as described in reference [40] is reviewed and several automorphic forms constructed from representations of $E_{n+1}(\mathbb{R})$ not discussed in [40] are analysed to test their compatibility with type IIB string theory.

Chapter nine forms the basis for reference [30]. We investigate the behaviour of the higher order terms and their automorphic forms in the $d = 10 - n$ dimensional effective action of type IIA/B string theory and M-theory in a unified manner through the E_{11} formulation of supergravity. In particular, we examine the behaviour of a higher derivative term and its automorphic form in the $d = 10 - n$ dimensional weak coupling limit, the type IIA volume limit, the type IIB volume limit,

the M-theory volume limit and the decompactification of a single dimension limit. It is found that each of these limits is associated with a specific node or nodes of the E_{n+1} Dynkin diagram.
As well as containing useful formulae and identities the appendix provides a review of the essential Lie group and Lie algebra theory used throughout this thesis and introduces the theory of non-linear realisations.

2 Supergravity

In its infancy supergravity was hoped to be a field theory of gravity free of ultraviolet divergences. However, these days supergravity is often considered to be the low-energy effective theory, at second order in derivatives, of type IIA/B string theory in ten dimensions and M-theory in eleven dimensions. The fundamental idea behind supergravity is to take a field theory of gravity and add supersymmetry, which is a symmetry between the bosonic and fermionic sectors of the theory, that we will elucidate on shortly. The addition of supersymmetry to a field theory powerfully constrains the field content of the theory and provides a tool to investigate non-perturbative features of the theory by protecting certain quantities, allowing one to extrapolate from a weak coupling regime to a strong coupling regime. In this chapter we provide a brief review of type IIA/B supergravity and eleven dimensional supergravity, as well as describing the correspondence between type IIA/B supergravity and the two derivative terms in the effective actions of type IIA/B string theory.

2.1 d=11 Supergravity

Eleven dimensional supergravity is a $d = 11$ massless field theory containing gravity with an eleven dimensional supersymmetry algebra. The bosonic fields of a relativistic massless field theory in $d = 11$ must transform in representations of the $SO(9)$ little group of the full $d = 11$ Poincare group. The field content consists of the graviton, the gravitino and a rank three antisymmetric gauge field. The graviton, which transforms as a symmetric, traceless, two index tensor representation of $SO(9)$ and group theoretically lies in the $\mathbf{44}$ of $SO(9)$, carries 44 bosonic degrees of freedom. The fermionic counterpart of the graviton is the spin $\frac{3}{2}$ gravitino ψ_μ, which carries both $SO(9)$ vector and spinor indices, although the spinor indices have been suppressed, and transforms under the direct product $\mathbf{16} \times \mathbf{9}$ of $SO(9)$ representations. The direct product $\mathbf{16} \times \mathbf{9}$ of $SO(9)$ representations decomposes as $\mathbf{16} \times \mathbf{9} = \mathbf{128} + \mathbf{16}$, when supplemented by the traceless condition $\gamma^\mu \psi_\mu = 0$ the propagating fermionic degrees of freedom are reduced by 16 to 128. A supersymmetric field theory requires an equal number of fermionic and bosonic degrees of freedom, so far we have 44 bosonic degrees of freedom from the graviton and 128 fermionic degrees of freedom associated with the gravitino. The remaining bosonic degrees of freedom are found in the rank three antisymmetric gauge field A, which carries three antisymmetrised $SO(9)$ vector indices, it thus transforms as the antisymmetric product of three representations of the $\mathbf{9}$ of $SO(9)$ and has $\frac{9!}{6!3!} = 84$ degrees of freedom.

The eleven dimensional supergravity Lagrangian was originally constructed in [8] and may be written

$$\begin{aligned}\mathcal{L}_{11} = & \frac{1}{4\kappa_{11}^2} eR - \frac{1}{2.4!} eF_{\mu_1\mu_2\mu_3\mu_4} F^{\mu_1\mu_2\mu_3\mu_4} - \frac{1}{2} e\overline{\psi}_\mu \gamma^{\mu\nu\sigma} D_\nu \left(\frac{1}{2} \left(\Omega + \hat{\Omega} \right) \right) \psi_\sigma \\ & - \frac{1}{192} e\kappa_{11} \left(\overline{\psi}_{\mu_1} \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} \psi_{\mu_2} + 12 \overline{\psi}^{\mu_3} \gamma^{\mu_4\mu_5} \psi^{\mu_6} \right) \left(F_{\mu_3\mu_4\mu_5\mu_6} + \hat{F}_{\mu_3\mu_4\mu_5\mu_6} \right) \\ & + \frac{2}{12^4} \kappa_{11} \epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8\mu_9\mu_{10}\mu_{11}} F_{\mu_1\mu_2\mu_3\mu_4} F_{\mu_5\mu_6\mu_7\mu_8} A_{\mu_9\mu_{10}\mu_{11}},\end{aligned} \tag{2.1}$$

where eleven dimensional coordinate indices are denoted by Greek letters and the corresponding tangent space indices are Greek letters with an overline, e is the determinant of the eleven dimensional vielbein $e_\mu{}^{\overline{\nu}}$ and

$$\begin{aligned} F_{\mu_1\mu_2\mu_3\mu_4} &= 4\partial_{[\mu_1} A_{\mu_2\mu_3\mu_4]}, \\ \hat{F}_{\mu_1\mu_2\mu_3\mu_4} &= F_{\mu_1\mu_2\mu_3\mu_4} + 3\kappa_{11} \overline{\psi}_{[\mu_1} \gamma_{\mu_2\mu_3} \psi_{\mu_4}, \\ \hat{\Omega}_{\mu\overline{\nu}\overline{\sigma}} &= \omega_{\mu\overline{\nu}\overline{\sigma}} - \frac{1}{2}(\bar{\psi}_\nu \Gamma_{\overline{\sigma}} \psi_{\overline{\nu}} - \bar{\psi}_\nu \Gamma_{\overline{\nu}} \psi_{\overline{\sigma}} + \bar{\psi}_{\overline{\sigma}} \Gamma_\mu \psi_{\overline{\nu}}), \\ \tilde{\Omega}_{\mu\overline{\nu}\overline{\sigma}} &= \hat{\Omega}_{\mu\overline{\nu}\overline{\sigma}} + \frac{1}{4} \bar{\psi}_\nu \Gamma^{\nu\lambda}_{\mu\overline{\nu}\overline{\sigma}} \psi_\lambda, \\ \omega_{\mu\overline{\nu}\overline{\sigma}} &= \frac{1}{2}(e_{\overline{\nu}}{}^\rho \partial_\mu e_{\rho\overline{\sigma}} - e_{\overline{\sigma}}{}^\rho \partial_\mu e_{\rho\overline{\nu}}) - \frac{1}{2}(e_{\overline{\nu}}{}^\rho \partial_\rho e_{\mu\overline{\sigma}} - e_{\overline{\sigma}}{}^\rho \partial_\rho e_{\mu\overline{\nu}}) \\ &\quad - \frac{1}{2}(e_{\overline{\nu}}{}^\lambda e_{\overline{\sigma}}{}^\rho \partial_\lambda e_{\rho\overline{\alpha}} - e_{\overline{\sigma}}{}^\lambda e_{\overline{\nu}}{}^\rho \partial_\lambda e_{\rho\overline{\alpha}}) e_\mu{}^{\overline{\alpha}}. \end{aligned} \tag{2.2}$$

The action, constructed from the above Lagrangian, possesses general coordinate invariance, local Lorentz invariance and is invariant under the gauge transformation $A_{\mu_1\mu_2\mu_3} \to A_{\mu_1\mu_2\mu_3} + 3\partial_{[\mu_1}\Lambda_{\mu_2\mu_3]}$ where $\Lambda_{\mu_1\mu_2}$ are the components of a two-form. Being a supersymmetric theory, it is also invariant under the following supersymmetry transformations

$$\begin{aligned} \delta e_\mu{}^{\overline{\nu}} &= \kappa_{11} \bar{\epsilon} \gamma^\mu \psi_\mu, \\ \delta A_{\mu_1\mu_2\mu_3} &= -\frac{3}{2} \bar{\epsilon} \gamma_{[\mu_1\mu_2} \psi_{\mu_3]}, \\ \delta\psi_\mu &= -\frac{1}{\kappa_{11}} D_\mu \left(\hat{\Omega} \right) \epsilon + \frac{1}{144} \left(\gamma_\mu^{\nu_1\nu_2\nu_3\nu_4} - 8\delta_\mu^{\nu_1} \gamma^{\nu_2\nu_3\nu_4} \right) \hat{F}_{\nu_1\nu_2\nu_3\nu_4} \epsilon. \end{aligned} \tag{2.3}$$

2.2 Type IIA Supergravity

Type IIA supergravity is a ten dimensional maximally supersymmetric field theory containing gravity. The type IIA supergravity theory was originally constructed by dimensional reduction of the unique eleven dimensional supergravity theory, we will postpone this method of constructing the theory until chapter four and instead give the derivation of the type IIA field content from a group theoretical perspective. The massless fields in a $d = 10$ relativistic field theory are required to transform in representations of $SO(8)$. The triality symmetry of $SO(8)$ gives three highest weight representations of dimension eight, one of them is the vector representation of $SO(8)$, denoted $\mathbf{8_v}$ while the remaining two are spinor representations, denoted $\mathbf{8_s}$ and $\mathbf{8_c}$. Type IIA supergravity is obtained by taking the direct product of two supermultiplets $(\mathbf{8_v} + \mathbf{8_s})$ and $(\mathbf{8_v} + \mathbf{8_c})$ which are formed from spinor representations $\mathbf{8_s}$ and $\mathbf{8_c}$ of opposite chirality. If one lets $(\mathbf{8_v} + \mathbf{8_s}) \times (\mathbf{8_v} + \mathbf{8_c}) = \mathbf{b} + \mathbf{f}$, where $\mathbf{b}$ are the $SO(8)$ representations of the type IIA bosonic fields and $\mathbf{f}$ are the $SO(8)$ representations of the fermionic fields, we find

$$\mathbf{b} = \mathbf{1} + \mathbf{8_v} + \mathbf{28} + \mathbf{35_v} + \mathbf{56_v} \tag{2.4}$$

and

$$\mathbf{f} = \mathbf{8_s} + \mathbf{8_c} + \mathbf{56_s} + \mathbf{56_c}. \tag{2.5}$$

For the bosonic fields, the $\mathbf{1}$ corresponds to a scalar, which one identifies as the type IIA dilaton. The $\mathbf{35_v}$ is the symmetric traceless rank 2 tensor representation, which is the $d = 10$ graviton. The $\mathbf{8_v}$ is the vector representation, which provides eight propagating bosonic degrees of freedom and may be realised by a one-form gauge field. The $\mathbf{28}$ and $\mathbf{56_v}$ are antisymmetric tensor representations of $SO(8)$ of rank 2 and 3, respectively. The $\mathbf{28}$ may be realised by a two-form gauge field while the $\mathbf{56_v}$ may be realised by a three-form gauge field. For the fermionic fields we have two gravitinos of opposite chirality which transform in the $\mathbf{56_s}$ and $\mathbf{56_c}$ and two opposite chirality dilatinos that transform in the $\mathbf{8_s}$ and $\mathbf{8_c}$. Thus we find 128 propagating degrees of freedom in both the bosonic and fermionic sectors.

As mentioned earlier, the $d = 10$ type IIA supergravity action may be derived by dimensional reduction of the $d = 11$ supergravity action, which we will perform in chapter three. For now, we will state the bosonic part of the Lagrangian that gives the type IIA action, suppressing the fermionic terms which may also be derived through dimensional reduction of the $d = 11$ supergravity action but are considerably more complicated than the bosonic terms. In Einstein frame, the bosonic part of the type IIA supergravity action is

$$\begin{aligned}\mathcal{L}_{IIA} = \frac{1}{\kappa_{10}^2}\bigg(&\frac{1}{2}eR - \frac{1}{12}ee^{\frac{\phi}{2}}\hat{F}_{\mu_1\mu_2\mu_3\mu_4}\hat{F}^{\mu_1\mu_2\mu_3\mu_4} - \frac{1}{3}ee^{-\phi}F_{\mu_1\mu_2\mu_3}F^{\mu_1\mu_2\mu_3} \\ &-\frac{1}{4}ee^{\frac{3}{2}\phi}F_{\mu_1\mu_2}F^{\mu_1\mu_2} - \frac{1}{2}e\partial_\mu\phi\partial^\mu\phi + \frac{1}{288}\epsilon^{\mu_1\ldots\mu_{10}}F_{\mu_1\mu_2\mu_3\mu_4}F_{\mu_5\mu_6\mu_7\mu_8}A_{\mu_9\mu_{10}}\bigg),\end{aligned} \tag{2.6}$$

where greek indices are ten dimensional coordinate indices while greek indices with an overline are tangent space indices, e is the determinant of the vielbein with components $e_\mu{}^{\overline{\nu}}$, ϕ is the type IIA supergravity dilaton and

$$\begin{aligned}F_{\mu_1\mu_2} &= 2\partial_{[\mu_1}A_{\mu_2]}, \\ F_{\mu_1\mu_2\mu_3} &= 3\partial_{[\mu_1}A_{\mu_2\mu_3]}, \\ \hat{F}_{\mu_1\mu_2\mu_3\mu_4} &= 4\left(\partial_{[\mu_1}A_{\mu_2\mu_3\mu_4]} + A_{[\mu_1}F_{\mu_2\mu_3\mu_4]}\right).\end{aligned} \tag{2.7}$$

The type IIA supergravity action possesses an $GL(1,\mathbb{R})$ symmetry. In Einstein frame, one finds that the transformations

$$\begin{aligned} \phi &\to \phi + c, \\ A_\mu &\to e^{-\frac{3}{4}c} A_\mu, \\ A_{\mu_1\mu_2} &\to e^{\frac{1}{2}c} A_{\mu_1\mu_2}, \\ A_{\mu_1\mu_2\mu_3} &\to e^{-\frac{1}{4}c} A_{\mu_1\mu_2\mu_3}. \end{aligned} \tag{2.8}$$

leave the type IIA supergravity action invariant.

2.3 Type IIB Supergravity

Unlike the $d = 10$ type IIA supergravity theory, one cannot obtain $d = 10$ type IIB supergravity by dimensional reduction of eleven dimensional supergravity. However, in deriving the field content of type IIA supergravity, in the previous section, we constructed an $N = 2$ supermultiplet by taking a direct product of $N = 1$ supermultiplets containing spinor representations of opposite chirality, if we choose to construct the $d = 10$, $N = 2$ supergravity theory by taking the direct product of two $N = 1$ supermultiplets containing spinor representations of the same chirality we obtain type IIB supergravity.

The direct product of 2 multiplets of the form $\mathbf{8_v} + \mathbf{8_c}$ decomposes as $(\mathbf{8_v} + \mathbf{8_c}) \times (\mathbf{8_v} + \mathbf{8_c}) = \mathbf{b_{IIB}} + \mathbf{f_{IIB}}$ where

$$\mathbf{b_{IIB}} = \mathbf{1} + \mathbf{1} + \mathbf{28} + \mathbf{28} + \mathbf{35_v} + \mathbf{35_c} \tag{2.9}$$

and

$$\mathbf{f_{IIB}} = \mathbf{8_s} + \mathbf{8_s} + \mathbf{56_s} + \mathbf{56_s}. \tag{2.10}$$

The bosonic type IIB supergravity fields lie in representations contained in $\mathbf{b_{IIB}}$ while their fermionic counterparts lie in representations contained in $\mathbf{f_{IIB}}$. In the bosonic sector of the type IIB supergravity theory, one may identify the two fields lying in the trivial representations of $SO(8)$ as the type IIB supergravity dilaton and axion, the propagating degrees of freedom of the graviton are again contained in the $\mathbf{35_v}$, while the two $\mathbf{28}$ representations are attributable to two 2-form gauge fields. The bosonic field corresponding to the $\mathbf{35_c}$ are realised by a four-form gauge field, however, a four-form gauge field in ten dimensions has $\frac{8!}{4!4!} = 70$ independent components, therefore to give the correct propagating degrees of freedom a self duality condition must be imposed on the five-form field strength constructed out of the four-form gauge field. Moreover, the self duality condition must be imposed on the five-form field strength at the level of the equations of motion, rather than on the five-form field strength in the type IIB supergravity action.

The fermionic type IIB supergravity fields lie in spinor representations of $SO(8)$ contained in $\mathbf{f_{IIA}}$. For the type IIB supergravity theory there are two gravitinos with the same chirality, each transforming in the $\mathbf{56_s}$ of $SO(8)$, and two dilatinos of the same chirality transforming in the $\mathbf{8_s}$ of

$SO(8)$. So, as expected, we find 128 propagating bosonic degrees of freedom and 128 propagating fermionic degrees of freedom.

The bosonic and fermionic sectors of type IIB supergravity both contain massless fields that transform under representations of $SO(8)$ that are degenerate, for the bosonic sector we have two copies of the trivial representation and the **28** of $SO(8)$, while in the fermionic sector we find two copies of the $\mathbf{56_s}$ and $\mathbf{8_s}$ spinor representations of $SO(8)$. This hints at another symmetry of the IIB theory and indeed the type IIB supergravity possesses an $SL(2,\mathbb{R})$ non-compact symmetry, that we will describe in greater depth in chapter four.

The self-duality condition on the five-form field strength constructed out of the four-form gauge field presents an obstruction to formulating the type IIB supergravity action. If one takes the standard two derivative term for the five-form field strength $F_{\mu_1\mu_2\mu_3\mu_4\mu_5} = 5\partial_{[\mu_1}A_{\mu_2\mu_3\mu_4\mu_5]}$ in the type IIB supergravity action, given by

$$\int d^{10}x\sqrt{-g}F_{\mu_1\mu_2\mu_3\mu_4\mu_5}F^{\mu_1\mu_2\mu_3\mu_4\mu_5}, \tag{2.11}$$

and attempts to implement the self-duality condition $F_{\mu_1\mu_2\mu_3\mu_4\mu_5} = *F_{\mu_1\mu_2\mu_3\mu_4\mu_5}$ by substituting this constraint into the action, one finds that the standard two derivative term for the five-form field strength vanishes. It is possible to write a covariant action for the type IIB theory that contains the correct degrees of freedom by following the PST approach [49] but we will not make use of this formulation. Instead, we will use an action for the type IIB supergravity theory in ten dimensions that gives the correct equations of motion after imposing the self-duality condition as an additional constraint on these equations. This action, in string frame, takes the form

$$\begin{aligned}\mathcal{L}_{IIB} = &\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2.3!}H_{\mu_1\mu_2\mu_3}H^{\mu_1\mu_2\mu_3}\right)\\ &-\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left(\frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2.3!}\tilde{F}_{\mu_1\mu_2\mu_3}\tilde{F}^{\mu_1\mu_2\mu_3} - \frac{1}{4.5!}\tilde{F}_{(5)\ \mu_1\mu_2\mu_3\mu_4\mu_5}\tilde{F}_{(5)}^{\mu_1\mu_2\mu_3\mu_4\mu_5}\right)\\ &-\frac{1}{4\kappa^2}\int A_{(4)}\wedge H_3\wedge F_3,\end{aligned} \tag{2.12}$$

where χ is the type IIB supergravity axion, $H_{\mu_1\mu_2\mu_3}$ and $F_{\mu_1\mu_2\mu_3}$ are the components of the three form field strengths $H = dB_{(2)}$ and $F = dC_{(2)}$ constructed out of the two-form gauge fields $B_{(2)}$ and $C_{(2)}$, while $F_{(5)\mu_1\mu_2\mu_3\mu_4\mu_5}$ are the components of the five-form field strength $F = dA_{(4)}$ constructed out of the four-form gauge fields. The field strengths $\tilde{F}$ and $\tilde{F}_{(5)}$ appearing in the action are the components of the gauge-invariant definitions, given by

$$\begin{aligned}\tilde{F} &= F_{(3)} - \chi H_{(3)},\\ \tilde{F}_{(5)} &= F_{(5)} - \frac{1}{2}C_{(2)}\wedge H_{(3)} + \frac{1}{2}B_{(2)}\wedge F_{(3)}.\end{aligned} \tag{2.13}$$

The self-duality condition on the five-form field strength $\tilde{F}_{(5)}$ is

$$\tilde{F}_{(5)} = *\tilde{F}_{(5)}, \tag{2.14}$$

this must be implemented on the equations of motion that follow from varying the action in (2.12).

Although, the type IIB supergravity theory possesses a non-compact $SL(2,\mathbb{R})$ symmetry, the above action does not make it manifest. In chapter three we will give a manifestly $SL(2,\mathbb{R})$ invariant action for the type IIB supergravity theory and convert this action to a formulation transforming non-linearly with respect to $SL(2,\mathbb{R})$.

2.4 Supergravity as a Low-energy Effective Theory

In this section we will show that the massless spectrum of type IIA/B string theory is identical to that of type IIA/B supergravity and thus one may identify the type IIA/B supergravity action with the two derivative part of the effective action of type IIA/B string theory.

Type IIA and type IIB are closed, supersymmetric, string theories that are non-chiral and chiral, respectively. In addition, open strings may be present if they end on Dp-branes. For type IIA string theory the set of stable Dp-branes possess p odd, while type IIB string theory contains stable Dp-branes for p even. The stable Dp-branes in type IIA and type IIB string theory couple electrically or magnetically to the corresponding $p+1$ form R-R gauge field.

M-theory is the strong coupling limit $g_s \to \infty$ of type IIA string theory [52, 53]. In this limit the closed string and the stable D4-brane of the type IIA theory become an M2-brane and an M5-brane in the eleven dimensional theory that couple electrically and magnetically to a three form gauge field. M-theory does not admit a well-defined perturbative expansion and is not nearly as well understood as type IIA/B string theory, however the two derivative terms in the effective action of M-theory form the familiar eleven dimensional supergravity action in (2.1). It is hoped that the higher derivative terms in the M-theory effective action will shed further light on the full quantum theory. In the rest of this thesis, any reference to M-theory should be understood as the eleven dimensional supergravity theory found in the low-energy limit.

2.4.1 Type IIA/B String Theory

We will review the derivation of the massless spectrum of type IIA/B string theory via the RNS formalism of the superstring, which, although not manifestly space-time supersymmetric, is suitable for this purpose. For a more complete treatment of the RNS superstring the reader is referred to references [50, 51].

Our starting point is the $d = 10$ world sheet action for the superstring in conformal gauge

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left(\partial_\alpha X_\mu \partial^\alpha X^\mu + \overline{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu \right), \tag{2.15}$$

where $X_\mu(\sigma,\tau)$ are the world sheet target space coordinates transforming under $D = 10$ Lorentz transformations and world sheet diffeomorphisms, $\psi_\mu(\sigma,\tau)$ are anticommuting world sheet spinor fields lying in the Majorana representation of the $d = 2$ Clifford algebra that also transform as vectors under $D = 10$ Lorentz transformations and γ^α are gamma matrices satisfying a $d = 2$ Clifford algebra

$$\left\{\gamma^\alpha, \gamma^\beta\right\} = 2\eta^{\alpha\beta}. \tag{2.16}$$

A Majorana representation of the $d = 2$ Clifford algebra is given by taking

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.17}$$

The RNS action possesses global world sheet supersymmetry. The supersymmetry transformations that leave the action invariant, up to a total derivative that vanishes under certain boundary conditions, are

$$\begin{aligned} \delta X^\mu &= \overline{\epsilon}\psi^\mu, \\ \delta\psi^\mu &= \gamma^\alpha \partial_\alpha X^\mu \epsilon, \end{aligned} \tag{2.18}$$

where ϵ is an infinitesimal Grassmanian Majorana spinor.

Prior to the quantisation process, it is wise to adopt world sheet light cone coordinates, in which $\sigma^\pm = \tau \pm \sigma$. Defining the conjugate of the $d = 2$ spinor field ψ^μ by

$$\overline{\psi}^\mu = (\psi^\mu)^\dagger \gamma^3, \tag{2.19}$$

where $\gamma^3 = i\gamma^0$. The world sheet action in light cone coordinates is

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \left(2\partial_\alpha X_\mu \partial^\alpha X^\mu + i\left(\psi_-^\mu \partial_+ \psi_{-\mu} + \psi_+^\mu \partial_- \psi_{+\mu} \right) \right). \tag{2.20}$$

The classical equations of motion for X^μ derived from the RNS string action written in world sheet coordinates are

$$\partial_+ \partial_- X^\mu = 0. \tag{2.21}$$

One should note that the boundary conditions for the world sheet bosonic fields in the closed string case is the periodicity condition $X^\mu(\sigma,\tau) = X^\mu(\sigma + 2\pi, \tau)$ that gives rise to left and right moving sectors in the general solution to the X^μ equations of motion. For the fermionic world sheet fields

ψ^μ the equations of motion are

$$\begin{aligned} \partial_+ \psi^\mu_- &= 0, \\ \partial_- \psi^\mu_+ &= 0. \end{aligned} \tag{2.22}$$

However, in varying the action with respect to $\psi^\mu_\pm$ one also obtains boundary terms that must vanish,

$$\delta S \sim \int d\tau \left(\psi^\mu_+ \delta\psi_{+\mu} - \psi^\mu_- \delta\psi_{-\mu} \right). \tag{2.23}$$

For the closed string, there are two conditions under which these boundary terms vanish

$$\psi_\pm (\sigma) = \pm \psi_\pm (\sigma + 2\pi). \tag{2.24}$$

The periodic choice, associated with the positive sign are known as Ramond R boundary conditions, while the antiperiodic choice, associated with the negative sign are Neveu-Schwarz (NS) boundary conditions. One may choose Ramond or Neveu-Schwarz boundary conditions in the left and right moving sectors ψ_- and ψ_+ separately, this results in the closed string containing four distinct sectors. The sector in which left movers and right movers satisfy Ramond boundary conditions, denoted R-R, gives rise to spacetime bosons, as does the NS-NS sector, while the NS-R and R-NS sectors contain spacetime fermions.

To quantise the RNS string one may introduce canonical commutation and anticommutation relations for the bosonic and fermionic fields world sheet fields, respectively. The oscillator modes in the general solution to the equations of motion are then upgraded to operators on Hilbert spaces in both the NS and R sectors. The R and NS sector Hilbert spaces are constructed, in the usual way, by acting on the ground state in each Hilbert space with the corresponding R or NS raising operators, in addition to the bosonic raising operators. The Hilbert spaces constructed in this way contain negative norm states that are eliminated via the super-Virasoro conditions in the critical dimension $D = 10$, leaving a spectrum of physical states. However, this spectrum contains a tachyon in the NS sector. Moreover there is no fermionic counterpart to the tachyon in the spectrum, so space-time supersymmetry appears broken. To remove the tachyon and regain a supersymmetric physical state spectrum one may perform a GSO projection. This involves defining the G-parity operator that acts on states to return whether the number of world sheet fermion excitations, in that state, is an odd number or an even number.

To implement the GSO projection one removes all states with a negative G-parity in the NS sector. The NS sector then only consists of states constructed from an odd number of NS sector raising operators denoted b^μ_r, where $r \in \mathbb{Z} + \frac{1}{2}$. Since the tachyon appears as the ground state of the RNS string in the NS sector it is removed by the GSO projection, and, in addition, leaves a supersymmetric spectrum of RNS string physical states. In the R sector, the G-parity operator similarly returns whether the number of world sheet fermion excitations is an odd or an even number, and one may choose to remove either states with positive or negative G-parity from the R sector physical state spectrum. Since the R sector ground state is a massless spinor, choosing to remove states with a positive or negative G-parity results in projecting out those states with positive or negative chirality from the R sector physical state spectrum.

After the GSO projection, the massless spectrum of the closed string in the RNS formulation for both the type IIA and type IIB string theories is composed of four sectors, corresponding to the possible choices of fermionic boundary conditions for the left and right moving sectors of the closed string. The massless (ground) state in the R sector is an eight component spinor with positive chirality $|+\rangle_R$ or negative chirality $|-\rangle_R$, while the massless (ground) state in the NS sector is an eight component vector in either the left moving $\tilde{b}^i_{-\frac{1}{2}}|0\rangle_{NS}$ or right moving sectors $b^i_{-\frac{1}{2}}|0\rangle_{NS}$, where $i = 1, ...8$, of the RNS string. The four sectors may then be constructed by taking the tensor product of a massless state in the left moving sector with a massless state in the right moving sector.

The type IIA theory may be identified by taking the left moving and right moving R sector ground states to have opposite chirality. The resulting four massless physical state sectors of the type IIA theory are then

$$|-\rangle_R \otimes |+\rangle_R, \tag{2.25}$$

$$\tilde{b}^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes |+\rangle_R, \tag{2.26}$$

$$|-\rangle_R \otimes b^i_{-\frac{1}{2}}|0\rangle_{NS}, \tag{2.27}$$

$$\tilde{b}^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes b^j_{-\frac{1}{2}}|0\rangle_{NS}. \tag{2.28}$$

In the R-R sector we see that we have the tensor product of two Majorana-Weyl spinors of opposite chirality, the R-R sector states are therefore Bosons. The decomposition of the tensor product gives a one-form gauge field and a three-form gauge field. The NS-NS sector is the tensor product of two eight component vectors, so the NS-NS sector also consists of Bosonic states. The tensor product of the two eight component vectors decomposes as an antisymmetric two-form gauge field, a symmetric traceless rank-two tensor and a singlet state. The graviton may be identified as the symmetric traceless rank-two tensor while the singlet state is the type IIA dilaton. The NS-R and R-NS sectors are the tensor products of an eight component vector and a Majorana-Weyl spinor (where the NS-R and R-NS Majorana-Weyl spinors are of opposite chirality), and are therefore fermions. The decomposition of the tensor product gives a gravitino and a dilatino, providing 56 states and 8 states, respectively. Thus, one finds that the R-R and NS-NS sectors contain 128 bosonic states, while the NS-R and R-NS sectors contain 128 fermionic states.

The type IIB theory may be identified by taking the left moving and right moving R sector ground states to have the same chirality. The resulting four massless physical state sectors of the type IIB theory are then

$$|+\rangle_R \otimes |+\rangle_R, \tag{2.29}$$

$$\tilde{b}^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes |+\rangle_R, \tag{2.30}$$

$$|+\rangle_R \otimes b^i_{-\frac{1}{2}}|0\rangle_{NS}, \tag{2.31}$$

$$\tilde{b}^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes b^j_{-\frac{1}{2}}|0\rangle_{NS}. \tag{2.32}$$

The NS-NS sector is identical to the type IIA theory NS-NS sector, one finds an antisymmetric two-form gauge field, a symmetric traceless rank-two tensor, the graviton, and a singlet state, the type IIB dilaton. The R-R sector is the tensor product of two Majorana-Weyl spinors of the same chirality, the decomposition of which gives a singlet state (the type IIB axion), a two-form gauge field and a four-form gauge field with a self duality condition that must be imposed on its field strength. The NS-R and R-NS sectors are the tensor products of an eight component vector and a Majorana-Weyl spinor of the same chirality in each sector. The decomposition of the tensor product gives a gravitino and a dilatino, providing 56 states and 8 states, respectively. So the massless spectrum of the type IIB RNS string also contains 128 bosonic states in the NS-NS and R-R sectors and 128 fermionic states in the R-NS and NS-R sectors.

The massless field content of the type IIA and type IIB RNS string are identical to their supergravity counterpart's field content. Furthermore, although it is not manifest in the RNS formulation, the massless spectrum of the type IIA and type IIB string possesses $N = 2$ spacetime supersymmetry. In the low-energy limit $\alpha' p_\mu p^\mu \to 0$ the massive modes of the type IIA/B string vanish, leaving the massless spectrum. By virtue of the fact that there are only two $N = 2$ supersymmetry theories in ten dimensions one would expect that the two derivative terms in the effective action of the type IIA and type IIB string, containing only the massless spectrum, would be very similar to type IIA and type IIB supergravity action in ten dimensions, indeed this is the case.

To make contact with the NS-NS sector of the type IIA/B supergravity action one starts from the string sigma model action with bosonic world sheet fields X^μ coupled to both the background metric $g_{\mu\nu}$ and a rank 2 antisymmetric tensor $A_{\mu\nu}$ that are functions of the target space coordinates and, in addition, add a term that is linear in the two dimensional world sheet scalar curvature $R^{(2)}$ and coupled to the dilaton ϕ. Insisting on conformal invariance in the string sigma model action is equivalent to the vanishing of the beta functions of the background fields $B_{\mu\nu}$, $g_{\mu\nu}$ and ϕ, this gives a set of equations that these fields must satisfy. One may interpret the equations arising from the vanishing of the beta functions of the background fields as the equations of motion of the background fields themselves. However, the NS-NS sector of the type IIA/B effective action that delivers equations of motion that coincide with the equations arising from the vanishing of the beta functions of the background fields appears multiplied by a factor of $e^{-2<\phi_{A\backslash B}>}$, where $<\phi_{A\backslash B}>$ is the expectation value of the type IIA/B dilaton, this is the so-called string frame action. Rescaling the vielbein by taking $e^{s}{}_{\mu}^{\overline{\nu}} = e^{\frac{1}{4}\phi} e_{\mu}{}^{\nu}$ puts the type IIA/B classical supergravity action in string frame, where it may then be compared to the type IIA/B effective action. Comparing the coupling constants in the type IIA/B effective action and the classical supergravity action, one finds $2\kappa_{10}^2 = (2\pi)^7 (\alpha')^2 e^{2<\phi>}$. So, at lowest order in α', one finds that the classical type IIA and type IIB supergravity actions give equations of motion that coincide with the equations of motion derived from the vanishing of the beta functions of the background fields in the string sigma model action. It therefore follows that the two derivative NS-NS sector part of the type IIA/B effective action is given by the corresponding type IIA/B supergravity action, with the classical supergravity coupling constant replaced by the string coupling constants α' and $g_s = e^{<\phi>}$. One should note that the R-R sector fields do not appear in the sigma model string action but it is well known that they may be similarly related to their classical supergravity counterparts, but appear multiplied by a factor of $e^{2\phi}$ relative to the NS-NS sector terms, signalling their appearance at one order beyond the NS-NS fields in a perturbative expansion of g_s.

As described above, the effective action of type IIA and type IIB string theory at order α' may be identified with the type IIA and type IIB classical supergravity action. However, the complete effective action contains an infinite series of higher derivative corrections. The rest of this thesis is concerned with investigating and identifying the higher derivative terms and their coefficient functions that transform as automorphic forms that appear in the complete effective action of type IIA string theory, type IIB string theory and M-theory. For completeness, we will end this chapter by giving the low-energy effective actions of type IIA and type IIB string theory, with their appropriate stringy parameters. We are primarily concerned with the bosonic fields in the massless spectrum therefore any fermionic terms in the effective action are suppressed in the rest of this thesis.

2.4.2 Type IIA and M-theory Low-energy Effective Actions

The type IIA low-energy effective action, in string frame, at second order in derivatives is

$$\begin{aligned} S_{IIA} = \frac{1}{(2\pi)^7 l_s^8} \int d^{10}x\sqrt{-g_s}e^{-2\phi} \Big(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2\cdot 3!} H_{\mu_1\mu_2\mu_3}H^{\mu_1\mu_2\mu_3} - \frac{1}{2\cdot 2!} e^{2\phi} F_{\mu_1\mu_2}F^{\mu_1\mu_2} \\ - \frac{1}{2\cdot 4!} e^{2\phi} \tilde{F}_{\mu_1\mu_2\mu_3\mu_4}\tilde{F}^{\mu_1\mu_2\mu_3\mu_4} \Big) - \frac{1}{2\cdot(2\pi)^7 l_s^8} \int B_2 \wedge F_4 \wedge F_4, \end{aligned} \tag{2.33}$$

where $F_{\mu_1\mu_2}$ are the components of the two-form field strength $F_2 = dA_1$, similarly $H_{\mu_1\mu_2\mu_3}$ and $\tilde{F}^{\mu_1\mu_2\mu_3\mu_4}$ are the components of the three and four-form field strengths defined by $H_3 = dB_2$ and $F_4 = dC_3 + A_1 \wedge H_3$. Note that Einstein frame, in which the curvature R does not appear multiplied by a factor of the dilaton ϕ, is related to string frame by $g_{E\mu\nu} = e^{-\frac{\phi}{2}} g_{s\mu\nu}$, where $g_{E\mu\nu}$

is the Einstein frame metric and $g_{s\mu\nu}$ is the string frame metric.

In chapter three we will derive the two derivative terms in the type IIA effective action by dimensional reduction of the two derivative terms in the M-theory effective action written in the form

$$S_M = \frac{1}{2\kappa_{11}^2} \int d^{11}x\sqrt{-g}\left(R - \frac{1}{2\cdot 4!}F_{\mu_1\mu_2\mu_3\mu_4}F^{\mu_1\mu_2\mu_3\mu_4}\right) - \frac{1}{12\kappa_{11}^2}\int A_3 \wedge F_4 \wedge F_4 \tag{2.34}$$

where R is the eleven dimensional scalar curvature, A_3 is the three form gauge field, $F_4 = dA_3$ is the four-form field strength constructed from the three form gauge field and $F_{\mu_1\mu_2\mu_3\mu_4}$ are the components of the four-form field strength.

2.4.3 Type IIB Low-energy Effective Action

The type IIB low-energy effective action, in string frame, at second order in derivatives is

$$\begin{aligned} S_{IIB} = \frac{1}{(2\pi)^7 l_s^8} \int d^{10}x\sqrt{-g_s}e^{-2\phi}\Bigg(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2\cdot 3!}H_{\mu_1\mu_2\mu_3}H^{\mu_1\mu_2\mu_3} \\ -\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2\cdot 3!}\tilde{F}_{3\mu_1\mu_2\mu_3}\tilde{F}^{\mu_1\mu_2\mu_3} - \frac{1}{4\cdot 5!}\tilde{F}_{5\mu_1\mu_2\mu_3\mu_4\mu_5}\tilde{F}_5^{\mu_1\mu_2\mu_3\mu_4\mu_5}\Bigg) \\ - \frac{1}{2\cdot(2\pi)^7 l_s^8}\int C_4 \wedge H_3 \wedge F_3 \end{aligned} \tag{2.35}$$

where χ is the type IIB axion, C_4 is the R-R four-form gauge field, $H_{\mu_1\mu_2\mu_3}$, $F_{\mu_1\mu_2\mu_3}$ and $\tilde{F}_{5\mu_1\mu_2\mu_3\mu_4\mu_5}$ are the components of the three form field strength constructed from the NS-NS two-form gauge field $H = dB_2$, R-R two-form gauge field $F_3 = dA_2$ and the self dual five-form field strength $\tilde{F}_5 = dC_4 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$. The five-form field strength self duality condition that must be imposed on the equations of motion that result from varying the type IIB action is $\tilde{F}_5 = *\tilde{F}_5$. Although not manifest in this formulation, type IIB supergravity possesses an $SL(2,\mathbb{R})$ symmetry that is broken to an $SL(2,\mathbb{Z})$ subgroup for the full type IIB string theory, this is explained further in chapter three.

3 Compactification and Dualities

The $E_{n+1}(\mathbb{Z})$ duality symmetries we will use to investigate the higher order terms in the effective action of type IIA/B string theory and M-theory appear after compactifying type IIA/B string theory on an n torus and M-theory on an $n+1$ torus. Although, it has been speculated that these symmetries may be present in the uncompactified theory [65]. This chapter describes the compactification process and reviews the duality symmetries of type IIA/B string theory compactified on an n torus and M-theory compactified on an $n+1$ torus. The reader is referred to reference [54] for an introduction to the Kaluza-Klein reduction of supergravity theories.

3.1 Compactification on S^1

Consider a D dimensional relativistic scalar field theory in Minkowski space, with a scalar field $\phi(x^\mu)$ that is a function of the background coordinates x^M, $M = 1, ...D$, and an action, schematically given by

$$\int d^d x \partial_M \phi \partial^M \phi^*, \tag{3.1}$$

where ϕ^* is the complex conjugate of ϕ. If we compactify one of the coordinates on a circle S^1 of radius r and let the coordinate of the circle be denoted by z while the remaining $D-1$ coordinates are denoted by μ, $\mu = 1, ..., D-1$, one now requires that $\phi(x^\mu, z)$ be periodic in z. As such, we may perform a Fourier decomposition of the scalar field $\phi(x^\mu, z)$ over the periodic coordinate

$$\phi(x^\mu, z) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) e^{i\frac{nz}{r}}. \tag{3.2}$$

Note that $\phi_n^* = \phi_{-n}$. The derivative ∂_M decomposes into a derivative over the non-compact directions μ and the compact direction z, it then takes the form $\partial_M = \partial_\mu + \partial_z$. Substituting the Fourier decomposition of ϕ into the action gives

$$\int d^d x \partial_M \phi \partial^M \phi^* = 2\pi r \int d^{d-1} x \sum_{n=-\infty}^{\infty} \left(\partial_\mu \phi_n \partial^\mu \phi_n + \frac{n^2}{r^2} \phi_n \phi_{-n} \right). \tag{3.3}$$

Calculating the equations of motion arising from the compactified action one finds

$$\partial_\mu \partial^\mu \phi_n + \frac{n^2}{r^2} \phi_n = 0, \tag{3.4}$$

which is the wave equation for a free scalar field of mass $m^2 = \frac{n^2}{r^2}$. For small r, all Fourier modes with $n \neq 0$ have such a high mass that they can effectively be ignored when the energy scale is of the order $E << \frac{1}{r}$. This will usually be the case in this thesis where we are principally concerned with the effective action and therefore the massless sector of type IIA/B string theory and M-theory. Compactified theories for which the massive Fourier modes are neglected are referred to as dimensionally reduced rather than compactified. One should note that the fields in a dimensionally reduced theory are independent of the compact coordinate z.

So far we have seen that dimensional reduction of a free (single) scalar field theory results in D dimensions results in a free (single) scalar field theory in $D-1$ dimensions after truncation to the massless sector. However, the case we are really concerned with is the dimensional reduction of the effective action of type IIA/B string theory and M-theory. The field content of these theories consists of the metric, antisymmetric gauge fields and possibly scalars, so we would like to know how these behave when our theory is dimensionally reduced from D to $D-1$ dimensions.

Upon dimensional reduction from D to $D-1$ dimensions the metric $\hat{g}_{MN}$ splits into a symmetric rank 2 tensor in the non-compact directions μ, a one-form field, with one non-compact index and one compact index, and a scalar, carrying only compact indices, in $D-1$ dimensions. This split allows one to take the following compactification ansatz for the D dimensional metric in terms of the $D-1$ dimensional fields,

$$d\hat{s}^2 = e^{2\alpha\rho} ds^2 + e^{2\beta\rho} \left(dz + A_\mu dx^\mu\right) \left(dz + A_\nu dx^\nu\right), \tag{3.5}$$

where we have adopted notation such that D dimensional fields carry a $\hat{}$ and α, β are left as non-zero arbitrary constants for now. From this ansatz one may identify the components of the D dimensional metric in terms of the $D-1$ dimensional fields, we see that

$$\hat{g}_{\mu\nu} = e^{2\alpha\rho} g_{\mu\nu} + e^{2\beta\rho} A_\mu A_\nu, \tag{3.6}$$

$$\hat{g}_{\mu z} = e^{2\beta\rho} A_\mu, \tag{3.7}$$

$$\hat{g}_{zz} = e^{2\beta\rho}. \tag{3.8}$$

The D dimensional gauge fields compactify in a similar way to the metric. A k-form gauge field $\hat{A}$ in D dimensions splits into a k-form gauge field with components $A_{\mu_1 \ldots \mu_k}$, in k non-compact directions and a $(k-1)$-form gauge field with components $A_{\mu_1 \ldots \mu_{k-1} z}$ in $k-1$ non-compact directions and a single compact direction, so we have

$$\begin{aligned} \hat{A} &= \frac{1}{k!} \hat{A}_{M_1 \ldots M_k} dx^{M_1} \wedge \ldots \wedge dx^{M_k} \\ &= \frac{1}{k!} A_{\mu_1 \ldots \mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} + \frac{1}{(k-1)!} A_{\mu_1 \ldots \mu_{k-1} z} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{k-1}} \wedge dz, \end{aligned} \tag{3.9}$$

where the expected coefficient of the $(k-1)$-form field $\frac{1}{(k-1)!}$ arises from the k ways of choosing the compact coordinate. Since the $D-1$ dimensional fields are taken to be independent of the compact coordinate z, a $(k+1)$-form field strength $\hat{F} = d\hat{A}$ constructed from a k-form gauge field

A splits into a $(k+1)$-form field strength and a k-form field strength in $D-1$ dimensions,

$$\hat{F} = dA^{(k)} + dA^{(k-1)} \wedge dz, \tag{3.10}$$

where $A^{(k)}$ and $A^{(k-1)}$ are the $D-1$ dimensional k and $(k-1)$-form gauge fields in (3.9).

As we have seen, a D dimensional scalar gives rise to an infinite set of scalars upon compactification. However, after truncation to the massless spectrum the D dimensional scalar gives a single $D-1$ dimensional scalar.

3.2 Compactification on an n Torus

One may compactify more than one dimension by repeatedly compactifying one of the non-compact coordinates on a circle S^1 using the procedure described in the previous section and, at each step, treating the $D-1$ fields arising from the compactification process as the higher dimensional fields for the next compactification. Repeating this process n times is equivalent to compactifying a D dimensional theory on an n torus $T^n = S^1 \times S^1 ... \times S^1$. However, our approach to compactifying a D dimensional theory on an n torus is to perform the reduction in a single step. Upon compactification, the D non-compact coordinates, which we shall denote by capital Roman indices $M, N, ...$, split into $D-n$ non-compact coordinates, with indices denoted by greek letters $\mu, \nu, ...$ and n compact coordinates with indices $i, j, ...$ written in lower case Roman letters. Assuming the radii of the torus are small, one may again truncate to the massless sector, ignoring the Kaluza-Klein modes, at low energies.

To compactify D dimensional type IIA/B string theory or M-theory on an n torus we will take the following ansatz for the D dimensional metric,

$$d\hat{s}_D^2 = e^{2\alpha\rho} g_{\mu\nu} dx^\mu dx^\nu + e^{2\beta\rho} G_{ij} \left(dx^i + A^i_\mu dx^\mu\right) \left(dx^j + A^j_\nu dx^\nu\right), \tag{3.11}$$

where G_{ij} are the components of the internal metric on the n-torus which satisfies $det(G) = 1$, A^i_μ are the components of the $d = D-n$ dimensional graviphotons and the constants α and β are defined by

$$\alpha^2 = \frac{n}{2(D-2)(d-2)}, \tag{3.12}$$

$$\beta = -\frac{D-2}{n}\alpha. \tag{3.13}$$

Our definition of α and β ensures that the $d = D-n$ dimensional theory is given in Einstein frame and, as we will see from the dimensional reduction of the scalar curvature, sets the coefficient of the two derivative kinetic term for the volume modulus ρ to the usual factor of $\frac{1}{2}$. From the compactification ansatz (3.11), one may identify the higher dimensional components of the metric

$\hat{g}$ in terms of the $D-n$ dimensional fields, we see

$$\hat{g}_{\mu\nu} = e^{2\alpha\rho} g_{\mu\nu} + e^{2\beta\rho} g_{ij} A^i_\mu A^j_\nu, \tag{3.14}$$

$$\hat{g}_{\mu j} = e^{2\beta\rho} G_{ij} A^j_\mu, \tag{3.15}$$

$$\hat{g}_{ij} = e^{2\beta\rho} G_{ij}. \tag{3.16}$$

We will also require the components of the inverse D dimensional metric $\hat{g}^{-1}$ as a function of the $D-n$ dimensional fields, these are given by

$$\hat{g}^{\mu\nu} = e^{-2\alpha\rho} g^{\mu\nu}, \tag{3.17}$$

$$\hat{g}^{\mu j} = -e^{-2\alpha\rho} g^{\mu\nu} A^j_\nu, \tag{3.18}$$

$$\hat{g}^{ij} = e^{-2\beta\rho} G^{ij} + e^{-2\alpha\rho} g^{\mu\nu} A^i_\mu A^j_\nu. \tag{3.19}$$

In addition, we may expand the components of the D dimensional metric $\hat{g}_{MN}$ and inverse metric $\hat{g}^{MN}$ in terms of a choice of vielbein $\hat{e}_M{}^{\overline{N}}$ and inverse vielbein $\hat{e}^M{}_{\overline{N}}$. For a metric on a Lorentzian manifold, $\hat{g}_{MN} = \hat{e}_M{}^{\overline{K}} \hat{e}_N{}^{\overline{L}} \eta_{\overline{KL}}$ and $\hat{g}^{MN} = \hat{e}^M{}_{\overline{K}} \hat{e}^N{}_{\overline{L}} \eta^{\overline{KL}}$, where overlined indices are local $SO(1, D-1)$ indices. The D dimensional vielbein and inverse vielbein may then be expressed as a function of the $D-n$ dimensional fields. For the D dimensional vielbein one finds,

$$\begin{aligned}
\hat{e}_\mu{}^{\overline{\nu}} &= e^{\alpha\rho} e_\mu{}^{\overline{\nu}}, \\
\hat{e}_\mu{}^{\overline{j}} &= e^{\beta\rho} A^i_\mu e_i{}^{\overline{j}}, \\
\hat{e}_i{}^{\overline{\nu}} &= 0, \\
\hat{e}_i{}^{\overline{j}} &= e^{\beta\rho} e_i{}^{\overline{j}},
\end{aligned} \tag{3.20}$$

where $e_\mu{}^{\overline{\nu}}$ is the vielbein of the $D-n$ dimensional metric, $e_i{}^{\overline{j}}$ is the vielbein of the metric on the n-torus and overlined greek indices are local $SO(1, D-1-n)$ indices while overlined lower case Roman indices are local $SO(n)$ indices. The components of the inverse D dimensional vielbein as a function of the $D-n$ dimensional fields are

$$\begin{aligned}
\hat{e}^\mu{}_{\overline{\nu}} &= e^{-\alpha\rho} e^\mu{}_{\overline{\nu}}, \\
\hat{e}^\mu{}_{\overline{j}} &= 0, \\
\hat{e}^i{}_{\overline{\nu}} &= -e^{-\alpha\rho} A^i_\mu e^\mu{}_{\overline{\nu}}, \\
\hat{e}^i{}_{\overline{j}} &= e^{-\beta\rho} e^i{}_{\overline{j}},
\end{aligned} \tag{3.21}$$

where $e^{\mu}{}_{\overline{\nu}}$ is the vielbein of the inverse $D-n$ dimensional metric and $e_i{}^{\overline{j}}$ is the vielbein of the inverse metric on the n-torus.

3.2.1 Compactification of the Curvature

The effective actions of type IIA/B string theory and M-theory contain gravity. The classical contribution appears in the supergravity action, in the form of the Einstein-Hilbert term

$$\int d^D x \sqrt{-\hat{g}}\hat{R}, \tag{3.22}$$

where $\hat{R}$ is the D dimensional scalar curvature. Under dimensional reduction on an n torus, the determinant of the D dimensional metric gives

$$\sqrt{-\hat{g}} = e^{((D-n)\alpha+n\beta)\rho}\sqrt{-g} = e^{2\alpha\rho}\sqrt{-g}, \tag{3.23}$$

where we have used the expression for β in (3.13). The calculation of the D dimensional scalar curvature as a function of the $D-n$ dimensional fields is more involved. To begin with, we will define the vielbein frame

$$\begin{aligned} \hat{e}^{\overline{\nu}} &= e^{\alpha\rho} e_{\mu}{}^{\overline{\nu}} dx^{\mu}, \\ \hat{e}^{\overline{i}} &= e^{\beta\rho} e_j{}^{\overline{i}} \left(dx^j + A^j_{\mu} dx^{\mu}\right). \end{aligned} \tag{3.24}$$

The D dimensional components of the spin connection $\hat{\omega}$ are then found from setting the torsion to zero in Cartan's first structure equation. This yields

$$\begin{aligned} d\hat{e}^{\overline{\nu}} &= -\hat{\omega}^{\overline{\nu}}{}_{\mu} \wedge \hat{e}^{\overline{\mu}} - \omega^{\overline{\nu}}{}_{\overline{i}} \wedge e^{\overline{i}} \\ d\hat{e}^{\overline{i}} &= -\hat{\omega}^{\overline{i}}{}_{\mu} \wedge \hat{e}^{\overline{\mu}} - \omega^{\overline{i}}{}_{\overline{j}} \wedge e^{\overline{j}}. \end{aligned} \tag{3.25}$$

Taking the derivative of the background space vielbein frame element $\hat{e}^{\overline{\nu}}$ we obtain

$$\begin{aligned} d\hat{e}^{\overline{\nu}} &= \alpha\partial_{\mu}\rho e^{\alpha\rho} dx^{\mu} \wedge dx^{\overline{\nu}} + e^{\alpha\rho} de^{\overline{\nu}} \\ &= \alpha\partial_{\mu}\rho dx^{\mu} \wedge e^{\overline{\nu}} - e^{\alpha\rho}\omega^{\nu}{}_{\overline{\lambda}} \wedge e^{\overline{\lambda}} \\ &= \alpha e^{-\alpha\rho}\partial_{\overline{\mu}}\rho e^{\overline{\mu}} \wedge e^{\overline{\nu}} - \omega^{\overline{\nu}}{}_{\overline{\lambda}} \wedge \hat{e}^{\overline{\lambda}}. \end{aligned} \tag{3.26}$$

While the derivative of the internal space vielbein frame element $\hat{e}^{\overline{i}}$ is

$$d\hat{e}^{\overline{i}} = \beta\left(\partial_{\overline{\nu}}\rho\right) e^{-\alpha\rho}\hat{e}^{\overline{\nu}} \wedge \hat{e}^{\overline{i}} + e^{-\alpha\rho} e^j{}_{\overline{k}} \left(\partial_{\overline{\nu}} e_j{}^{\overline{i}}\right) \hat{e}^{\overline{\nu}} \wedge \hat{e}^{\overline{k}} + \frac{1}{2} e^{(\beta-2\alpha)\rho} F^{\overline{i}}_{\overline{\mu\nu}} \hat{e}^{\overline{\nu}} \wedge \hat{e}^{\overline{\mu}} \tag{3.27}$$

where in the last line we have converted the coordinate indices to tangent space indices in the derivatives and field strength and relabelled indices. Splitting the $SL(n)/SO(n)$ Cartan form into symmetric S and antisymmetric Q parts under the Cartan involution τ defined in (B.42),

$$e_{\bar{j}}{}^{k}\partial_\mu e_k{}^{\bar{i}} \quad = \quad S_\mu{}^{\bar{i}}{}_{\bar{j}} + Q_\mu{}^{\bar{i}}{}_{\bar{j}}, \tag{3.28}$$

where $Q_\mu^{(\bar{i}\bar{j})} = S_\mu^{[\bar{i}\bar{j}]} = 0$, we have

$$d\hat{e}^{\bar{i}} = \beta\left(\partial_{\bar{\nu}}\rho\right)e^{-\alpha\rho}\hat{e}^{\bar{\nu}}\wedge\hat{e}^{\bar{i}} + e^{-\alpha\rho}\left(S_\nu{}^{\bar{i}}{}_{\bar{j}} + Q_\nu{}^{\bar{i}}{}_{\bar{j}}\right)\hat{e}^{\bar{\nu}}\wedge\hat{e}^{\bar{j}} + \frac{1}{2}e^{(\beta-2\alpha)\rho}F^{\bar{i}}_{\bar{\mu}\bar{\nu}}\hat{e}^{\bar{\nu}}\wedge\hat{e}^{\bar{\mu}}. \tag{3.29}$$

Thus to find the D dimensional spin connection in terms of the $d = D - n$ dimensional fields we must solve the equations

$$\begin{aligned}
-\hat{\omega}^{\bar{i}}{}_{\bar{\mu}}\wedge\hat{e}^{\bar{\mu}} - \hat{\omega}^{\bar{i}}{}_{\bar{j}}\wedge\hat{e}^{\bar{j}} &= \beta\left(\partial_{\bar{\nu}}\rho\right)e^{-\alpha\rho}\hat{e}^{\bar{\nu}}\wedge\hat{e}^{\bar{i}} + e^{-\alpha\rho}\left(S_{\bar{\nu}}{}^{\bar{i}}{}_{\bar{j}} + Q_{\bar{\nu}}{}^{\bar{i}}{}_{\bar{j}}\right)\hat{e}^{\bar{\nu}}\wedge\hat{e}^{\bar{j}} + \frac{1}{2}e^{(\beta-2\alpha)\rho}F^{\bar{i}}_{\bar{\mu}\bar{\nu}}\hat{e}^{\bar{\nu}}\wedge\hat{e}^{\bar{\mu}},\\
-\hat{\omega}^{\bar{\nu}}{}_{\bar{\mu}}\wedge\hat{e}^{\bar{\mu}} - \hat{\omega}^{\bar{\nu}}{}_{\bar{j}}\wedge\hat{e}^{\bar{j}} &= \alpha e^{-\alpha\rho}\left(\partial_{\bar{\mu}}\rho\right)e^{\bar{\mu}}\wedge e^{\bar{\nu}} - \omega^{\bar{\nu}}{}_{\bar{\lambda}}\wedge\hat{e}^{\bar{\lambda}},
\end{aligned} \tag{3.30}$$

along with the symmetry conditions $\omega_{\bar{\mu}\bar{\nu}} = -\omega_{\bar{\nu}\bar{\mu}}$, $\omega_{\bar{i}\bar{j}} = -\omega_{\bar{j}\bar{i}}$. This gives

$$\begin{aligned}
\hat{\omega}^{\bar{i}}{}_{\bar{j}} =& - e^{-\alpha\rho}Q_{\bar{\mu}}{}^{\bar{i}}{}_{\bar{j}}\hat{e}^{\bar{\mu}},\\
\hat{\omega}^{\bar{i}}{}_{\bar{\nu}} =& \beta\left(\partial_{\bar{\nu}}\rho\right)e^{-\alpha\rho}\hat{e}^{\bar{i}} + e^{-\alpha\rho}\left(S_{\bar{\nu}}{}^{\bar{i}}{}_{\bar{j}} + Q_{\bar{\nu}}{}^{\bar{i}}{}_{\bar{j}}\right)\hat{e}^{\bar{j}} + \frac{1}{2}e^{(\beta-2\alpha)\rho}F^{\bar{i}}_{\bar{\mu}\bar{\nu}}\hat{e}^{\bar{\mu}},
\end{aligned} \tag{3.31}$$

and, from substituting the second equation in (3.31) into the second equation in (3.30) we find

$$\hat{\omega}^{\bar{\mu}}{}_{\bar{\nu}} = \omega^{\bar{\mu}}{}_{\bar{\nu}} + \alpha\partial_{\bar{\nu}}\rho e^{-\alpha\rho}\hat{e}^{\bar{\mu}} - \alpha\partial^{\bar{\mu}}\rho e^{-\alpha\rho}\hat{e}_{\bar{\nu}} - \frac{1}{2}e^{(\beta-2\alpha)\rho}F_{\bar{i}}{}^{\bar{\mu}}{}_{\bar{\nu}}\hat{e}^{\bar{i}}. \tag{3.32}$$

The Riemann curvature terms can be calculated using Cartan's second structure equation,

$$\begin{aligned}
\hat{R}^{\bar{i}}{}_{\bar{j}} =& d\hat{\omega}^{\bar{i}}{}_{\bar{j}} + \hat{\omega}^{\bar{i}}{}_{\bar{\lambda}}\wedge\hat{\omega}^{\bar{\lambda}}{}_{\bar{j}} + \hat{\omega}^{\bar{i}}{}_{\bar{k}}\wedge\hat{\omega}^{\bar{k}}{}_{\bar{j}},\\
\hat{R}^{\bar{i}}{}_{\bar{\nu}} =& d\hat{\omega}^{\bar{i}}{}_{\bar{\nu}} + \hat{\omega}^{\bar{i}}{}_{\bar{\lambda}}\wedge\hat{\omega}^{\bar{\lambda}}{}_{\bar{\nu}} + \hat{\omega}^{\bar{i}}{}_{\bar{k}}\wedge\hat{\omega}^{\bar{k}}{}_{\bar{\nu}},\\
\hat{R}^{\bar{\mu}}{}_{\bar{\nu}} =& d\hat{\omega}^{\bar{\mu}}{}_{\bar{\nu}} + \hat{\omega}^{\bar{\mu}}{}_{\bar{\lambda}}\wedge\hat{\omega}^{\bar{\lambda}}{}_{\bar{\nu}} + \hat{\omega}^{\bar{\mu}}{}_{\bar{k}}\wedge\hat{\omega}^{\bar{k}}{}_{\bar{\nu}}.
\end{aligned} \tag{3.33}$$

The D dimensional curvature two-form components are related to the D dimensional Riemann curvature terms by $\hat{R}^M{}_K = \frac{1}{2}\hat{R}^M{}_{KLN}\hat{e}^L\wedge\hat{e}^N$. Expanding this out gives

$$\begin{aligned}
\hat{R}^{\bar{i}}{}_{\bar{j}} =& \frac{1}{2}\left(\hat{R}^{\bar{i}}{}_{\bar{j}\bar{k}\bar{l}}\hat{e}^{\bar{k}}\wedge\hat{e}^{\bar{l}} + 2\hat{R}^{\bar{i}}{}_{\bar{j}\bar{k}\bar{\mu}}\hat{e}^{\bar{k}}\wedge\hat{e}^{\bar{\mu}} + \hat{R}^{\bar{i}}{}_{\bar{j}\bar{\mu}\bar{\nu}}\hat{e}^{\bar{\mu}}\wedge\hat{e}^{\bar{\nu}}\right),\\
\hat{R}^{\bar{i}}{}_{\bar{\nu}} =& \frac{1}{2}\left(\hat{R}^{\bar{i}}{}_{\bar{\nu}\bar{k}\bar{l}}\hat{e}^{\bar{k}}\wedge\hat{e}^{\bar{l}} + 2\hat{R}^{\bar{i}}{}_{\bar{\nu}\bar{k}\bar{\mu}}\hat{e}^{\bar{k}}\wedge\hat{e}^{\bar{\mu}} + \hat{R}^{\bar{i}}{}_{\bar{\nu}\bar{\mu}\bar{\lambda}}\hat{e}^{\bar{\mu}}\wedge\hat{e}^{\bar{\lambda}}\right),\\
\hat{R}^{\bar{\mu}}{}_{\bar{\nu}} =& \frac{1}{2}\left(\hat{R}^{\bar{\mu}}{}_{\bar{\nu}\bar{k}\bar{l}}\hat{e}^{\bar{k}}\wedge\hat{e}^{\bar{l}} + 2\hat{R}^{\bar{\mu}}{}_{\bar{\nu}\bar{k}\bar{\lambda}}\hat{e}^{\bar{k}}\wedge\hat{e}^{\bar{\lambda}} + \hat{R}^{\bar{\mu}}{}_{\bar{\nu}\bar{\rho}\bar{\lambda}}\hat{e}^{\bar{\rho}}\wedge\hat{e}^{\bar{\lambda}}\right).
\end{aligned} \tag{3.34}$$

We may then calculate the D dimensional Riemann curvature components, which appear as the coefficients of the wedge product of D dimensional vielbein elements on the right hand side of (3.34), in terms of the $d = D - n$ dimensional fields by evaluating the right hand side of Cartan's second structure equation (3.33) and reading off the coefficients of the wedge products of the D dimensional vielbein elements, one finds

$$\begin{aligned}
\hat{R}_{\overline{klij}} &= e^{-2\alpha\rho}\left(S_{\overline{\nu il}}S^{\overline{\nu}}_{\overline{jk}} - S_{\overline{\nu jl}}S^{\overline{\nu}}_{\overline{ik}} + \beta\partial^{\overline{\nu}}\rho\left(S_{\overline{\nu il}}\delta_{\overline{kj}} - S_{\overline{\nu ik}}\delta_{\overline{lj}} + S_{\overline{\nu jk}}\delta_{\overline{li}}\right) - S_{\overline{\nu jl}}\delta_{\overline{ik}}\right. \\
&\quad \left. +\beta^2\partial_{\overline{mu}}\rho\partial^{\overline{mu}}\rho\left(\delta_{\overline{kj}}\delta_{\overline{li}} - \delta_{\overline{ki}}\delta_{\overline{lj}}\right)\right), \\
\hat{R}_{\overline{\mu kij}} &= \frac{1}{2}e^{(\beta-3\alpha)\rho}\left(S^{\overline{\nu}}_{\overline{ik}}F_{\overline{j\nu\mu}} - S^{\overline{\nu}}_{\overline{jk}}F_{\overline{i\nu\mu}} - \beta\partial^{\overline{\nu}}\rho F_{\overline{i\nu\mu}}\delta_{\overline{jk}} + \beta\partial^{\overline{\nu}}\rho F_{\overline{j\nu\mu}}\delta_{\overline{ik}}\right), \\
\hat{R}_{\overline{\mu\nu ij}} &= 2e^{-2\alpha\rho}\left(\nabla_{[\overline{\mu}}Q_{\overline{\nu}]\overline{ij}} + \left(Q_{[\overline{\mu}}Q_{\overline{\nu}]}\right)_{\overline{ij}} + \frac{1}{4}e^{2(\beta-\alpha)\rho}\left(F_{\overline{i}}F_{\overline{j}}\right)_{[\overline{\mu\nu}]}\right), \\
\hat{R}_{\overline{\mu\nu ij}} &= 2e^{-2\alpha\rho}\left(\left(2\alpha\beta - \beta^2\right)\partial_{\overline{\mu}}\rho\partial_{\overline{\nu}}\rho\delta_{\overline{ij}} - \beta\nabla_{\overline{\mu}}\partial_{\overline{\nu}}\rho\delta_{\overline{ij}} - \alpha\beta\partial_{\overline{\lambda}}\rho\partial^{\overline{\lambda}}\rho\eta_{\overline{\mu\nu}}\delta_{\overline{ij}}\right. \\
&\quad -\frac{1}{4}e^{2(\beta-\alpha)\rho}\left(F_{\overline{i}}F_{\overline{j}}\right)_{\overline{\nu\mu}} - \nabla_{\overline{\mu}}S_{\overline{\nu ij}} + (\alpha-\beta)\left(\partial_{\overline{\nu}}\rho S_{\overline{\mu ij}} + \partial_{\overline{\mu}}\rho S_{\overline{\nu ij}}\right) \\
&\quad \left. -\alpha\partial^{\overline{\lambda}}\rho S_{\overline{\lambda ij}}\eta_{\overline{\mu\nu}} - \left(S_{\overline{\mu}}S_{\overline{\nu}}\right)_{\overline{ij}} - \left[S_{\overline{\nu}}, Q_{\overline{\mu}}\right]_{\overline{ij}}\right), \\
\hat{R}_{\overline{\lambda\mu i\nu}} &= \frac{1}{2}e^{(\beta-3\alpha)\rho}\left(e_{\overline{i}j}\nabla_{\overline{\nu}}F^{j}_{\overline{\lambda\mu}} + (\beta-\alpha)\left(2\partial_{\overline{\nu}}\rho F_{\overline{i\lambda\mu}} + \partial_{\overline{\mu}}\rho F_{\overline{i\lambda\nu}} + \partial_{\overline{\lambda}}\rho F_{\overline{i\nu\mu}}\right)\right. \\
&\quad \left.\alpha\partial^{\overline{\sigma}}\rho\left(F_{\overline{i\lambda\sigma}}\eta_{\overline{\mu\nu}} - F_{\overline{i\mu\sigma}}\eta_{\overline{\lambda\nu}}\right) + 2S_{\overline{\nu ij}}F^{\overline{j}}_{\overline{\lambda\mu}} + S_{\overline{\mu ij}}F^{\overline{j}}_{\overline{\lambda\nu}} + S_{\overline{\lambda ij}}F^{\overline{j}}_{\overline{\mu\nu}}\right), \\
\hat{R}_{\overline{\lambda\sigma\mu\nu}} &= e^{-2\alpha\rho}R_{\overline{\lambda\sigma\mu\nu}} - \frac{1}{2}e^{2(\beta-2\alpha)\rho}F^{\overline{i}}_{\overline{\lambda\sigma}}F^{\overline{j}}_{\overline{\sigma\nu}}\delta_{\overline{ij}} - \frac{1}{4}e^{2(\beta-2\alpha)\rho}\left(F^{\overline{i}}_{\overline{\lambda\mu}}F^{\overline{j}}_{\overline{\sigma\nu}}\delta_{\overline{ij}} - F^{\overline{i}}_{\overline{\lambda\nu}}F^{\overline{j}}_{\overline{\sigma\mu}}\delta_{\overline{ij}}\right) \\
&\quad -\alpha e^{-2\alpha\rho}\left(\nabla_{\overline{\lambda}}\partial_{\overline{\mu}}\rho\eta_{\overline{\sigma\nu}} - \nabla_{\overline{\lambda}}\partial_{\overline{\nu}}\rho\eta_{\overline{\sigma\mu}} - \nabla_{\overline{\sigma}}\partial_{\overline{\mu}}\rho\eta_{\overline{\lambda\nu}} + \nabla_{\overline{\sigma}}\partial_{\overline{\nu}}\rho\eta_{\overline{\lambda\mu}}\right) \\
&\quad \alpha^2 e^{-2\alpha\rho}\left(\partial_{\overline{\lambda}}\rho\partial_{\overline{\mu}}\rho\eta_{\overline{\sigma\nu}} - \partial_{\overline{\lambda}}\rho\partial_{\overline{\nu}}\rho\eta_{\overline{\sigma\mu}} + \partial_{\overline{\sigma}}\rho\partial_{\overline{\nu}}\rho\eta_{\overline{\lambda\mu}} - \partial_{\overline{\sigma}}\rho\partial_{\overline{\mu}}\rho\eta_{\overline{\lambda\nu}}\right. \\
&\quad \left. +\partial_{\overline{\kappa}}\rho\partial^{\overline{\kappa}}\rho\left(\eta_{\overline{\lambda\nu}}\eta_{\overline{\sigma\mu}} - \eta_{\overline{\lambda\mu}}\eta_{\overline{\sigma\nu}}\right)\right).
\end{aligned} \tag{3.35}$$

The components of the D dimensional Ricci tensor, denoted $\hat{R}_{\overline{MN}}$, are calculated in the usual way, $\hat{R}_{\overline{MN}} = \hat{R}^{\overline{L}}{}_{\overline{MLN}}$. This leads to the D dimensional Ricci tensor components being given by

$$\begin{aligned}
\hat{R}_{\overline{ij}} &= e^{-2\alpha\rho}\left(-\beta\nabla^2\rho\delta_{\overline{ij}} - \left((d-2)\alpha\beta + n\beta^2\right)\partial_{\overline{mu}}\rho\partial^{\overline{\mu}}\rho\delta_{\overline{ij}} - \frac{1}{4}e^{2(\beta-\alpha)\rho}\left(F_{\overline{i\mu\nu}}F_{\overline{j\mu\nu}}\right)\right. \\
&\quad \left. -\nabla^{\overline{\mu}}S_{\overline{\mu ij}} - \left((d-2)\alpha + n\beta\right)\partial^{\overline{\mu}}\rho S_{\overline{\mu ij}}\right), \\
\hat{R}_{\overline{\mu i}} &= \frac{1}{2}e^{(\beta-3\alpha)\rho}\left(e_{\overline{i}j}\nabla^{\overline{\nu}}F^{j}_{\overline{\mu\nu}} + 2S^{\overline{\nu}}_{\overline{ij}}F^{\overline{j}}_{\overline{\mu\nu}} + \left((d-4)\alpha - \beta(n-4)\right)\partial^{\overline{\nu}}\rho F_{\overline{i\mu\nu}}\right), \\
\hat{R}_{\overline{\mu\nu}} &= e^{-2\alpha\rho}\left(R_{\overline{\mu\nu}} + \frac{1}{2}e^{2(\beta-\alpha)\rho}\left(F^{\overline{i}}F_{\overline{i}}\right)_{\overline{\mu\nu}} - \left(n\beta + (d-2)\alpha\right)\nabla_{\overline{\mu}}\partial_{\overline{\nu}}\rho\right. \\
&\quad \left((d-2)\alpha^2 + \left(2\alpha\beta - \beta^2\right)n\right)\partial_{\overline{\mu}}\rho\partial_{\overline{\nu}}\rho - \left((d-2)\alpha^2 + \alpha\beta n\right)\partial_{\lambda}\rho\partial^{\lambda}\rho\eta_{\overline{\mu\nu}} \\
&\quad \left. -\alpha\nabla^2\rho\eta_{\overline{\mu\nu}} - S^{\overline{j}}_{\overline{i\mu}}S^{\overline{i}}_{\overline{j\nu}}\right).
\end{aligned} \tag{3.36}$$

The D dimensional scalar curvature $\hat{R}$ is then found as a function of the $d = D - n$ dimensional fields by taking $\hat{R} = \eta^{\overline{MN}} \hat{R}_{\overline{MN}}$. From (3.36), we find

$$\begin{aligned}\hat{R} = e^{-2\alpha\rho} \Big(& R - \frac{1}{4} e^{2(\beta-\alpha)\rho} G_{ij} F^{i}_{\mu\nu} F^{j\mu\nu} - S^{j}_{\mu i} S^{\mu i}_{j} - \gamma^2 \partial_\mu \rho \partial^\mu \rho \\ & -2 \left(n\beta + (d-1)\alpha \right) \nabla^2 \rho \Big) ,\end{aligned} \tag{3.37}$$

where the constant γ is given by

$$\gamma^2 = (d-1)(d-2)\alpha^2 + (2dn - 4n)\alpha\beta + n(n+1)\beta^2. \tag{3.38}$$

The constants α and β given in equations (3.12) and (3.13) then imply $\gamma^2 = \frac{1}{2}$, to return the usual numerical factor for the two derivative term of the scalar field ρ in Einstein frame.

We are now in a position to evaluate the Einstein-Hilbert term after dimensional reduction on an n torus, from (3.23) and (3.37), we have

$$\begin{aligned}\int d^D x \sqrt{-\hat{g}} R = (2\pi)^n \int d^d x \sqrt{-g} \Big(& R - \frac{1}{4} e^{2(\beta-\alpha)\rho} G_{ij} F^{i}_{\mu\nu} F^{j\mu\nu} - S^{j}_{mi} S^{\mu i}_{j} - \gamma^2 \partial_\mu \rho \partial^\mu \rho \\ & -2 \left(n\beta + (d-1)\alpha \right) \nabla^2 \rho \Big) ,\end{aligned} \tag{3.39}$$

where the factor $(2\pi)^n$ arises from performing the integrals over the n compact coordinates which are trivial since the $d = D - n$ dimensional fields are taken to be independent of the compact coordinates. We see that the factor $e^{2\alpha\rho}$ arising from the dimensional reduction of the D dimensional invariant volume element $\sqrt{-\hat{g}}$, cancels with the factor of $e^{-2\alpha\rho}$ given by dimensional reduction of the D dimensional scalar curvature $\hat{R}$, leaving the $d = D - n$ Einstein-Hilbert action in Einstein frame. Note that $2\left(n\beta + (d-1)\alpha\right)\nabla^2\rho$ is a total derivative so may be neglected.

Prior to dimensional reduction, the Einstein-Hilbert action in D dimensions is invariant under D dimensional local coordinate transformations. After taking the n torus compactification ansatz (3.11) one finds a $d = D - n$ dimensional theory of gravity. The dimensionally reduced theory is invariant under d dimensional local coordinate transformations, as expected for a $d = D - n$ dimensional theory of gravity. However, from the action (3.39) one also finds the gauge symmetry $A^i_\nu \to A^i_\nu + \partial_\nu \Lambda^i$, where Λ^i are arbitrary scalar fields, in addition to the $SL(n,\mathbb{R})$ symmetry corresponding to the length preserving diffeomorphisms of the torus G and the $GL(1,\mathbb{R})$ shift symmetry of the volume modulus ρ which may be thought of as a global $SL(n,\mathbb{R}) \times GL(1,\mathbb{R}) \cong GL(n,\mathbb{R})$ symmetry. For the ten dimensional maximal supergravity theories dimensionally reduced on an n torus, or equivalently the eleven dimensional maximal supergravity theory on an $n+1$ torus, the $GL(n,\mathbb{R})$ global symmetry is enlarged to an $E_{n+1}(\mathbb{R})$ symmetry which is discussed in section 3.4.3.

3.2.2 Compactification of the Gauge Fields

The terms in the effective action of type IIA/B string theory and M-theory involving the gauge fields and their field strengths are either topological, such as the two derivative Chern-Simons term, and therefore do not depend on the background metric g, or appear contracted by the background metric g. Our methods for dimensionally reducing terms in the effective action involving the gauge field or field strengths will differ depending on whether the term is purely topological or involves the metric g. However, for both the topological terms and those involving the metric we must first examine the D dimensional k-form field compactified on an n torus.

When dimensionally reduced on an n-torus a D dimensional k-form gauge field $\hat{A}$ splits into a sum of gauge fields of degree k to $k-n$,

$$\hat{A} = \sum_{l=0}^{n} \frac{1}{l!(k-l)!} A_{\mu_1 \ldots \mu_{k-l} i_1 \ldots i_l} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{k-l}} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_l}, \tag{3.40}$$

where the coefficient $\frac{1}{l!(k-l)!}$ arises from the number of ways of choosing l compact indices from the coordinates of the k form field and the sum terminates at $l = k$ if $k < n$. By taking the D dimensional exterior derivative of $\hat{A}$ one finds that the D dimensional $k+1$ form field strength $\hat{F} = d\hat{A}$ constructed from the D dimensional k form gauge field $\hat{A}$ in (3.40), similarly splits into a sum of $D-n$ dimensional field strengths of degree k to $k-n$,

$$\hat{F} = d\hat{A} = \sum_{l=0}^{n} \frac{1}{l!} dA^{(k-l)}_{i_1 \ldots i_l} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_l}, \tag{3.41}$$

where $A^{(k-l)}_{i_1 \ldots i_l}$ are the $(k-l)$-form gauge fields arising from the dimensional reduction of $\hat{A}$ in (3.40) and the exterior derivative on the right hand side of equation (3.41) is a $d = D - n$ dimensional exterior derivative since the d dimensional fields are independent of the compact coordinates.

To dimensionally reduce the topological terms in the effective action of type IIA/B string theory and M-theory one may simply carry out the above dimensional reduction process for the gauge fields and their field strengths and then substitute the field strengths and gauge fields appearing in the D dimensional action for those expanded in terms of the lower dimensional fields. We will provide a demonstration of this technique at the end of this chapter when dimensionally reducing the eleven dimensional supergravity Chern-Simons term on S^1.

To perform the dimensional reduction of terms that involve field strengths contracted with the metric $\hat{g}$ in the effective action of type IIA/B string theory and M-theory we will first rewrite the action in differential form notation. A standard two derivative term consisting of field strengths contracted with the D dimensional metric g, in the effective action of type IIA/B string theory and M-theory takes the form,

$$\int d^D x \sqrt{-\hat{g}} \frac{1}{k!} \hat{F}_{M_1 \ldots M_k} \hat{F}^{M_1 \ldots M_k}, \tag{3.42}$$

where $\hat{F}$ are the components of a D dimensional k-form field strength and $M_1, ..., M_k$ are D dimensional coordinate indices. Now, in general, the exterior product of a k-form $B^{(k)}$ and the dual of a k-form $*C^{(k)}$ on a D dimensional manifold M with metric g, is given by

$$B^{(k)} \wedge *C^{(k)} = \frac{1}{k!} B^{(k)}_{M_1 \ldots M_k} C^{(k)\, M_1 \ldots M_k} \sqrt{-g} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^D. \tag{3.43}$$

So, the standard two derivative terms in the type IIA/B string theory and M-theory action consisting of D dimensional field strengths $\hat{F}$ contracted with the D dimensional metric $\hat{g}$, can be expressed as

$$\int \hat{F} \wedge *\hat{F} = \int \frac{1}{k!} \hat{F}^{(k)}_{M_1 \ldots M_k} \hat{F}^{(k)\, M_1 \ldots M_k} \sqrt{-\hat{g}} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^D, \tag{3.44}$$

where the integrand is now a D form and the integral is over the D dimensional manifold M. However, to further simplify the dimensional reduction calculation, one may convert the components of the field strengths $\hat{F}$ in (3.44) to tangent frame by expanding the D dimensional inverse metric as $\hat{g}^{MN} = e^M{}_{\overline{I}} e^N{}_{\overline{J}} \delta^{\overline{IJ}}$, we then have

$$\begin{aligned} \int \frac{1}{k!} \hat{F}^{(k)}_{M_1 \ldots M_k} \hat{F}^{(k)\, M_1 \ldots M_k} \sqrt{-\hat{g}} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^D = \int \frac{1}{k!} \hat{F}^{(k)}_{\overline{I}_1 \ldots \overline{I}_k} \hat{F}^{(k)}_{\overline{J}_1 \ldots \overline{J}_k} \eta^{\overline{I}_1 \overline{J}_1} \eta^{\overline{I}_2 \overline{J}_2} \ldots \eta^{\overline{I}_k \overline{J}_k} \\ \times \sqrt{-\hat{g}} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^D, \end{aligned} \tag{3.45}$$

where an overline denotes a tangent space index and η is the Minkowski metric. Upon dimensional reduction on an n torus the components of the D dimensional field strength $\hat{F}$ will be expressible in terms of $d = D - n$ dimensional fields. To find the D dimensional field strength tangent space components $\hat{F}_{I_1 \ldots I_k}$ in terms of the $d = D - n$ dimensional fields we first take the D dimensional exterior derivative of the dimensionally reduced $(k-1)$-form gauge field $\hat{A}$ from which $\hat{F}$ is constructed through $\hat{F} = d\hat{A}$. From (3.41) this gives

$$\hat{F} = \sum_{l=0}^{n} \frac{1}{l!} dA^{(k-l)}_{i_1 \ldots i_l} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_l}. \tag{3.46}$$

Converting to tangent frame through the use of the relations

$$dx^\mu = e^{-\alpha\rho} e_{\overline{\nu}}{}^{\mu} \hat{e}^{\overline{\nu}}, \tag{3.47}$$

$$dx^i = e^{-\beta\rho} e_{\overline{j}}{}^{i} \hat{e}^{\overline{j}} - e^{-\alpha\rho} A^i_\mu e_{\overline{\nu}}{}^{\mu} \hat{e}^{\nu}, \tag{3.48}$$

one finds that the dimensional reduction of an arbitrary k-form field strength may be written

$$\begin{aligned}\hat{F} = \sum_{r=0}^{n} e^{-((k-r)\alpha+r\beta)\rho} \sum_{l=r}^{n} \left(\frac{(-1)^{l(r+1)}}{l!\,(k-l)!} F_{\overline{\mu}_1 \ldots \overline{\mu}_{k-l} \overline{i}_1 \ldots \overline{i}_l} \prod_{m=1}^{l} A^{\overline{i}_m}_{\overline{\mu}_{k-l+m}} \right) \\ \times\, \hat{e}^{\overline{\mu}_1} \wedge \ldots \hat{e}^{\overline{\mu}_{k-r}} \wedge \hat{e}^{\overline{i}_1} \wedge \ldots \wedge \hat{e}^{\overline{i}_r},\end{aligned} \tag{3.49}$$

where our convention is to shuffle all internal tangent space basis elements $\hat{e}^{\overline{i}}$ to the right of all spacetime tangent space basis elements $\hat{e}^{\overline{\mu}}$ and shuffle all contractions between the graviphoton internal indices and the field strength indices to the right of all internal component indices contracted with internal basis elements by making use of the antisymmetry of the field strength indices. Expanding the D dimensional field strength $\hat{F}$ in tangent space we have,

$$\begin{aligned}\hat{F} &= \frac{1}{k!} \hat{F}_{\overline{M}_1 \ldots \overline{M}_k} \hat{e}^{\overline{M}_1} \wedge \ldots \wedge \hat{e}^{\overline{M}_k} \\ &= \sum_{r=0}^{n} \frac{1}{r!\,(k-r)!} \hat{F}_{\overline{\mu}_1 \ldots \overline{\mu}_{k-r} \overline{i}_1 \ldots \overline{i}_r} \wedge \hat{e}^{\overline{\mu}_1} \wedge \ldots \hat{e}^{\overline{\mu}_{k-r}} \wedge \hat{e}^{\overline{i}_1} \wedge \ldots \wedge \hat{e}^{\overline{i}_r}.\end{aligned} \tag{3.50}$$

Therefore, upon comparing the coefficients of the basis elements of (3.49) and (3.50) we see that

$$\hat{F}_{\overline{\mu}_1 \ldots \overline{\mu}_{k-r} \overline{i}_1 \ldots \overline{i}_r} = e^{-((k-r)\alpha+r\beta)\rho} \sum_{l=r}^{n} \left(\frac{(-1)^{l(r+1)}\, r!\,(k-r)!}{l!\,(k-l)!} F_{\overline{\mu}_1 \ldots \overline{\mu}_{k-l} \overline{i}_1 \ldots \overline{i}_l} \prod_{m=1}^{l} A^{\overline{i}_m}_{\overline{\mu}_{k-l+m}} \right). \tag{3.51}$$

Under dimensional reduction we then find

$$\begin{aligned}\int \hat{F} \wedge *\hat{F} = \sum_{r=0}^{n} \int \frac{1}{r!\,(k-r)!} \hat{F}^{(k-r)}_{\overline{\mu}_1 \ldots \overline{\mu}_{k-r} \overline{i}_1 \ldots \overline{i}_r} \hat{F}^{(k-r)}_{\overline{\nu}_1 \ldots \overline{\nu}_{k-r} \overline{j}_1 \ldots \overline{j}_r} \eta^{\overline{\mu}_1 \overline{\nu}_1} \eta^{\overline{\mu}_2 \overline{\nu}_2} \ldots \eta^{\overline{\mu}_{k-r} \overline{\nu}_{k-r}} \delta^{\overline{i}_1 \overline{j}_1} \ldots \delta^{\overline{i}_r \overline{j}_r} \\ \times\, e^{2\alpha\rho} \sqrt{-g} dx^1 \wedge dx^2 \wedge \ldots \wedge \wedge dx^D,\end{aligned} \tag{3.52}$$

where the components of the field strength $\hat{F}$ in tangent frame are given as a function of the $d = D - n$ fields by (3.51) and the factor of $e^{2\alpha\rho}\sqrt{-g}$ arises from evaluating $\sqrt{-\hat{g}}$.

Alternatively, for larger degree field strengths it is desirable to rewrite equations (3.49), (3.50) and (3.51) in differential form notation where it is easier to keep track of the numerical factors involved in the reduction. The dimensional reduction of an arbitrary field strength, equation (3.49), becomes

$$\hat{F} = \sum_{r=0}^{n} e^{-((k-r)\alpha+r\beta)\rho} \sum_{l=r}^{n} \left(\frac{(-1)^{l(r+1)}}{l!} F^{((k-l))}_{\overline{i}_1 \ldots \overline{i}_l} \prod_{m=1}^{l} \wedge A^{\overline{i}_m} \right) \wedge \hat{e}^{\overline{i}_1} \wedge \ldots \wedge \hat{e}^{\overline{i}_r} \tag{3.53}$$

where we have taken the usual definition of a k-form $F^{(k)} = \frac{1}{k!}F_{\mu_1...\mu_k}dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}$. The expansion of a D dimensional field strength in this notation takes the form

$$\hat{F} = \sum_{r=0}^{n} \frac{1}{r!} \hat{F}^{(k-r)}_{\bar{i}_1...\bar{i}_r} \wedge \hat{e}^{\bar{i}_1} \wedge ... \wedge \hat{e}^{i_r}. \tag{3.54}$$

One may then compare the coefficients of the internal space basis elements of equations (3.53) and (3.54) to find

$$\hat{F}^{(k-r)}_{\bar{i}_1...\bar{i}_r} = e^{-((k-r)\alpha+r\beta)\rho} \sum_{l=r}^{n} \left(\frac{(-1)^{l(r+1)}\, r!}{l!} F^{((k-l))}_{\bar{i}_1...\bar{i}_l} \prod_{m=1}^{l} \wedge A^{\bar{i}_m} \right). \tag{3.55}$$

The standard bilinear term in the field strengths found in the effective action of type IIA/B string theory and M-theory dimensionally reduced on an n torus may then be written

$$\int \hat{F} \wedge *\hat{F} = \sum_{r=0}^{n} \frac{1}{r!} \int e^{2\alpha\rho} \hat{F}^{(k-r)}_{\bar{i}_1...\bar{i}_r} \wedge *\hat{F}^{(k-r)\bar{i}_1...\bar{i}_r}, \tag{3.56}$$

where the inner product given by the $\wedge *$ operation on the right hand side of the above equation is with respect to the $d = D - n$ dimensional components of the field strengths.

3.3 Eleven Dimensional Supergravity Dimensionally Reduced on S^1

In this section we will give an example of the compactification process by dimensionally reducing eleven dimensional supergravity on a circle S^1. Dimensional reduction of the unique eleven dimensional supergravity theory on S^1 gives type IIA supergravity. As discussed in chapter two, the classical type IIA supergravity action is the two derivative sector of the effective action of type IIA string theory. In this thesis we are concerned with the higher derivative terms in the effective action of both type IIA string theory and M-theory, therefore it will be instructive to see how the two derivative terms in the effective action of type IIA string theory and M-theory are connected via dimensional reduction.

The bosonic sector of the classical eleven dimensional supergravity action, which acts as the two derivative part of the effective action of M-theory, is

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-\hat{g}} \left(\hat{R} - \frac{1}{2 \cdot 4!} \hat{F}^{(4)}{}_{M_1M_2M_3M_4} \hat{F}^{(4)M_1M_2M_3M_4} \right) - \frac{1}{12\kappa_{11}^2} \int \hat{C^{(3)}} \wedge \hat{F}^{(4)} \wedge \hat{F}^{(4)}. \tag{3.57}$$

where $\hat{C}^{(3)}$ is the eleven dimensional supergravity three form gauge field and $\hat{F}^{(4)}{}_{M_1M_2M_3M_4}$ are the components of the $D = 11$ four-form field strength $F = dC^3$. To keep track of the numerical factors involved in the reduction process, it is advisable to rewrite the eleven dimensional supergravity action in terms of eleven forms integrated over an eleven dimensional manifold. The eleven dimensional supergravity Lagrangian is then,

$$S = \frac{1}{2\kappa_{11}^2} \int \left(\hat{R} \wedge *1 - \frac{1}{2} \hat{F}^{(4)} \wedge *\hat{F}^{(4)} \right) - \frac{1}{12\kappa_{11}^2} \int \hat{C}^{(3)} \wedge \hat{F}^{(4)} \wedge \hat{F}^{(4)}, \tag{3.58}$$

where, in general, the exterior product of a k-form $B^{(k)}$ and the dual of a k-form $*C^{(k)}$ on a D dimensional manifold with metric g, is given by

$$B^{(k)} \wedge *C^{(k)} = \frac{1}{k!} B^{(k)}_{M_1 \ldots M_k} C^{(k)\,M_1 \ldots M_k} \sqrt{-g} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^D. \tag{3.59}$$

The compactification ansatz is our usual n-torus ansatz (3.11) with $n = 1$ and the single compact coordinate denoted by z,

$$d\hat{s}_{11}^2 = e^{2\alpha\rho} g_{\mu\nu} dx^\mu dx^\nu + e^{2\beta\rho} \left(dz + A_\mu dx^\mu \right) \left(dz + A_\nu dx^\nu \right). \tag{3.60}$$

We will return to the evaluation of the constants α and β later in the derivation. Dimensional reduction of the $D = 11$ Einstein-Hilbert term gives

$$\int \hat{R} \wedge *1 = \int \left(R \wedge *1 - \frac{1}{2} e^{2(\beta - \alpha)\rho} \tilde{F} \wedge *\tilde{F} - \frac{1}{2} d\rho \wedge *d\rho \right) \tag{3.61}$$

where $F^{(2)} = dA^{(1)}$ is the two-form field strength constructed from the graviphoton $A^{(1)}$ appearing in the compactification ansatz and we have rewritten the inner products between k-forms in the resulting contribution to the $d = 10$ Lagrangian as exterior products of the k-forms and their duals. Note that the Hodge dual on the left hand side of the above equation is with respect to the eleven dimensional background manifold while the Hodge dual on the right hand side is with respect to the ten dimensional background manifold.

The dimensional reduction of the three form gauge field $\hat{C}^{(3)}$ and its four-form field strength $\hat{F}^{(4)} = d\hat{C}^{(3)}$ proceeds as described in section 3.2.2. The eleven dimensional three form field strength $\hat{C}^{(3)}$ becomes

$$\hat{C}^{(3)} = C^{(3)} + C^{(2)} \wedge dz, \tag{3.62}$$

therefore upon dimensional reduction on S^1 the four-form field strength $\hat{F}^{(4)} = d\hat{C}^{(3)}$ is

$$\hat{F}^{(4)} = dC^{(3)} + dC^{(2)} \wedge dz. \tag{3.63}$$

We then find that dimensional reduction of the eleven dimensional Chern-Simons term gives

$$\frac{1}{12} \hat{C}^{(3)} \wedge \hat{F}^{(4)} \wedge \hat{F}^{(4)} = \frac{1}{4} F^{(4)} \wedge F^{(4)} \wedge C^{(2)}, \tag{3.64}$$

after integrating by parts and throwing away the boundary terms, so that the $d = 10$ three form gauge field $C^{(3)}$ always appears in the type IIA $d = 10$ Lagrangian in the form $F^{(4)} = dC^{(3)}$. To evaluate the dimensional reduction on S^1 of the bilinear terms in the four-form field strength contained in the M-theory effective action we will make use of the formulas in section 3.2.2. Since we are reducing a four-form field strength on S^1 we take $k = 4$ and $n = 1$, equation (3.55) then gives

$$\hat{F}^{(4)} = e^{-4\alpha\rho}\left(F^{(4)} - F^{(3)} \wedge A\right), \tag{3.65}$$

$$\hat{F}^{(3)}_{\bar{i}_1} = e^{-(3\alpha+\beta)\rho}F^{(3)}. \tag{3.66}$$

Substituting these into equation (3.56) one finds

$$\begin{aligned}\hat{F} \wedge *\hat{F} &= e^{-6\alpha\rho}\left(F^{(4)} - F^{(3)} \wedge A\right) \wedge *\left(F^{(4)} - F^{(3)} \wedge A^{(1)}\right) \\ &\quad + e^{-(4\alpha+2\beta)\rho}F^{(3)} \wedge *F^{(3)} \\ &= e^{-6\alpha\rho}\tilde{F}^{(4)} \wedge *\tilde{F}^{(4)} + e^{-(4\alpha+2\beta)\rho}F^{(3)} \wedge *F^{(3)},\end{aligned} \tag{3.67}$$

where, in the last line, we have defined $\tilde{F}^{(4)} = F^{(4)} - F^{(3)} \wedge A^{(1)}$. The two-form field strength in the dimensionally reduced action $F^{(2)} = dA^{(1)}$ is invariant under the gauge transformation $\delta A^{(1)} = d\Lambda^{(0)}$, similarly the three form field strength $F^{(3)}$ is invariant under the gauge transformation $\delta C^{(2)} = dA^{(1)}$, while the four-form field strength $\tilde{F}^{(4)}$ is invariant under the gauge transformation $\delta C^{(3)} = d\Lambda^{(2)}$. However, since the four-form field strength $\tilde{F}^{(4)}$ contains the one-form field $A^{(1)}$ without an exterior derivative acting on it, under the gauge transformation of the graviphoton $\delta A^{(1)} = d\Lambda^{(0)}$, the three form field must also transform as $\delta A^{(3)} = d\Lambda \wedge A^{(2)}$ to preserve the gauge invariance of $\tilde{F}^{(4)}$.

The last step in our derivation is to evaluate the constants α and β, from equations (3.12) and (3.13) with $d = 10$ and $n = 1$ we find

$$\alpha = -\frac{1}{12} \tag{3.68}$$

$$\beta = \frac{2}{3} \tag{3.69}$$

where we have taken the negative sign when evaluating α^2 in equation (3.12) to give the correct IIA dilaton factor, which we may now identify as α, multiplying the R-R and NS-NS fields, when transformed to string frame.

Collecting the dimensionally reduced terms we find that the two derivative sector of the type

IIA effective action is

$$S_{IIA} = \frac{1}{2\kappa_{10}^2}\int\left(R\wedge *1 - \frac{1}{2}\tilde{F}_{(4)}\wedge *\tilde{F}^{(4)} - \frac{1}{2}e^{-\phi}F_{(3)}\wedge *F^{(3)} - \frac{1}{2}e^{\frac{1}{2}\phi}F_{(2)}\wedge *F^{(2)} - \frac{1}{2}d\phi\wedge *d\phi\right)$$
$$- \int \frac{1}{4\kappa_{10}^2}F^{(4)}\wedge F^{(4)}\wedge C^{(2)} \tag{3.70}$$

where we have redefined the type IIA dilaton, $\phi = \rho$, in line with usual notation, and integrated over the compact coordinate z of the circle S^1 with radius r_{11}, the type IIA supergravity coupling in Einstein frame in ten dimensions is then given by $\kappa_{11}^2 = 2\pi r_{11}\kappa_{10}^2$.

3.4 Dualities

As discussed earlier, after dimensional reduction on an n torus the Einstein-Hilbert action possesses a global $GL(n,\mathbb{R})$ symmetry. For type IIA/B supergravity on an n torus or, equivalently, eleven dimensional supergravity on an $n+1$ torus, this symmetry is enhanced to an $E_{n+1}(\mathbb{R})$ global symmetry [8–11]. However, in the full quantum string theory and M-theory the global $E_{n+1}(\mathbb{R})$ symmetry is broken, due to the quantisation conditions on the brane charges, to an $E_{n+1}(\mathbb{Z})$ subgroup. This $E_{n+1}(\mathbb{Z})$ subgroup is the U-duality group of type IIA/B string theory and M-theory compactified on a torus to $d = 10 - n$ dimensions [14]. From a type IIB string theory point of view, U-duality may be thought of as being generated by T-duality [55], which relates type IIA string theory and type IIB string theory wrapped on a circle, and S-duality, which relates type IIB string theory in different coupling regimes. We will begin by reviewing T-duality and S-duality before discussing the U-duality group and in particular how the effective actions of dimensionally reduced type IIA/B string theory and M-theory are formulated as non-linear realisations of the U-duality group.

3.4.1 T-duality

Type IIA and type IIB string theory compactified on a circle S^1 of radius r are related by T-duality. Under compactification on a circle of S^1 the momentum p of the closed type IIA/B string along the compact dimension, is quantised in units of $\frac{K}{r}$ where $K \in \mathbb{Z}$ is the Kaluza-Klein excitation number. The winding number $W \in \mathbb{Z}$ is the number of times the string wraps around the circle. The mass M of the string is given by

$$M^2 = \frac{K^2}{r^2} + \frac{r^2W^2}{\alpha'^2} + 4\left(N_L + N_R - 2\right), \tag{3.71}$$

where N_L and N_R are the number operators of the left and right moving modes, respectively. The string is also required to satisfy the level matching condition

$$N_R - N_L = WK. \tag{3.72}$$

Interchanging the winding number W and the Kaluza-Klein excitation number K, denoted $K \leftrightarrow W$, while making the transformation $r \to \tilde{r} = \frac{\alpha'}{r}$ leaves the mass spectrum and level matching condition unchanged. Moreover, the type IIA string compactified on a cirle of radius r is related to the type IIB string compactified on a circle of radius $\tilde{r}$ by the T-duality relation $r\tilde{r} = \alpha'$. By comparing the coupling of the constants of the dimensionally reduced type IIA and type IIB NS-NS sector effective actions one then finds

$$g_{s(B)} = \frac{\sqrt{\alpha'}}{r} g_{s(A)}, \tag{3.73}$$

where $g_{s(A)}$ and $g_{s(B)}$ are the type IIA and type IIB string coupling constants respectively.

Considering a more general compactification of type IIA or type IIB string theory on a torus of dimension n, one finds that the generalisation of T-duality for a single compact dimension is an $SO(n,n;\mathbb{Z})$ symmetry. T-duality is a full symmetry of the interacting theory and acts on the NS-NS sectors and R-R sectors of type IIA/B string theory separately. The NS-NS and R-R sectors of the massless fields in the type IIA and type IIB theories will therefore transform separately under representations of the T-duality group $SO(n,n;\mathbb{Z})$. For the NS-NS sector, the components of the fields G_{ij}, B_{ij} in the compact directions $i,j = 1,...,n$ parameterise an $O(n,n;\mathbb{R})/\,(O(n;\mathbb{R}) \times (O(n;\mathbb{R})$ coset $\mathcal{G}$ which may be written,

$$\mathcal{G} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}, \tag{3.74}$$

where the components of $\mathcal{G}$ are expressed in $n \times n$ blocks. The generalised T-duality transformation equivalent to the inversion symmetry $r \to \tilde{r} = \frac{\alpha'}{r}$ for the case of a single compact dimension that leaves the mass spectrum and level matching conditions of the left moving and right moving modes of the string is then

$$W^i \leftrightarrow K_i, \quad \mathcal{G} \leftrightarrow \mathcal{G}^{-1}, \tag{3.75}$$

where $W^i \in \mathbb{Z}$ and $K_i \in \mathbb{Z}$ are the components of n vectors containing the winding numbers and the total momentum in the compact directions $i = 1,...,n$, respectively. In addition one has the shift symmetries

$$B_{ij} \to B_{ij} + N_{ij}, \tag{3.76}$$

while simultaneously taking

$$W_i \to W_i \tag{3.77}$$

and

$$K_i \to K_i + N_{ij} W^j, \tag{3.78}$$

where $N_{ij} \in \mathbb{Z}$ are the components of an antisymmetric matrix. These transformations may be realised through the group $O(n,n;\mathbb{Z})$ as

$$\mathcal{G} \to \mathcal{G}' = g_0 \mathcal{G} g_0^T \tag{3.79}$$

and

$$\begin{pmatrix} W \\ K \end{pmatrix} \to \begin{pmatrix} W' \\ K' \end{pmatrix} = g_0 \begin{pmatrix} W \\ K \end{pmatrix}, \tag{3.80}$$

where $g_0 \in O(n,n;\mathbb{Z})$. Since the full $O(n,n;\mathbb{Z})$ group contains inversion elements that reverse the chirality of the spinors, the T-duality group for type IIA/B string theory is the chirality preserving subgroup $SO(n,n;\mathbb{Z})$.

The R-R sector fields are similarly known to transform under the type IIA/B T-duality group $SO(n,n;\mathbb{Z})$. However, T-duality invariant formulations of the R-R sector fields have proven more difficult to construct. References [56–58] contain several approaches to treating the T-duality symmetry of the R-R sector fields.

3.4.2 S-duality

Type IIB supergravity in ten dimensions possesses a global $SL(2,\mathbb{R})$ symmetry [4]. However, in the full quantum theory the $SL(2,\mathbb{R})$ symmetry is broken to an $SL(2,\mathbb{Z})$ subgroup, which is the S-duality group of type IIB string theory in ten dimensions [14, 59–64]. S-duality is a strong-weak coupling duality that relates type IIB string theory to itself in different regimes of the string coupling g_s.

Before reviewing the $SL(2,\mathbb{Z})$ S-duality symmetry of the full theory we will rewrite the type IIB supergravity action with a global $SL(2,\mathbb{R})$ symmetry in non-linearly realised form as a means to illustrate the methods involved in constructing non-linear realisations that are used throughout the rest of this thesis and described in detail in appendix $B.3$.

The following type IIB action is written in standard $SL(2,\mathbb{R})$ covariant form and produces the correct equations of motion when supplemented by a self-duality condition for the five-form field strength. In Einstein frame, the action may be expressed as

$$\begin{aligned} S =& \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{12} H^T_{\mu_1\mu_2\mu_3} M H^{\mu_1\mu_2\mu_3} + \frac{1}{4} tr\left(\partial^\mu M \partial_\mu M^{-1}\right)\right) \\ &- \frac{1}{8\kappa^2} \left(\int d^{10}x \sqrt{-g} \frac{1}{5!} \tilde{F}_{\mu_1\mu_2\mu_3\mu_4\mu_5} \tilde{F}^{\mu_1\mu_2\mu_3\mu_4\mu_5} + \int \epsilon_{ab} C_4 \wedge H_3^a \wedge H_3^b \right), \end{aligned} \tag{3.81}$$

where a and b are $SL(2,\mathbb{R})$ doublet indices, $\tau = \chi + ie^{-\phi}$ is the axion-dilaton field and

$$M = e^{\phi}\begin{pmatrix} |\tau|^2 & -\chi \\ -\chi & 1 \end{pmatrix}. \tag{3.82}$$

The three form field strength H, with components $H_{\mu_1\mu_2\mu_3}$, is defined by $d\tilde{B}$ where

$$\tilde{B} = \begin{pmatrix} B \\ C \end{pmatrix}, \tag{3.83}$$

and B, C are two-forms in the NS-NS and R-R sectors respectively. In terms of the $SL(2,\mathbb{R})$ doublet indices, we have $H^1 = dB$ and $H^2 = dC$. The five-form field strength F is written in the form

$$\tilde{F} = F + \frac{1}{2}\epsilon_{ij}\tilde{B}^i \wedge H^j, \tag{3.84}$$

with components $\tilde{F}_{\mu_1\mu_2\mu_3\mu_4\mu_5}$. Note that $F = dC_4$, where C_4 is the R-R sector four-form. The self-duality condition, mentioned earlier, on the five-form field strength is $\tilde{F}_5 = *\tilde{F}_5$. The type IIB supergravity action in this form is invariant under the global $SL(2,\mathbb{R})$ symmetry group with transformations

$$\begin{aligned} \tau &\to \frac{a\tau + b}{c\tau + d}, \\ \tilde{B} &\to \Lambda\tilde{B}, \\ M &\to \left(\Lambda^{-1}\right)^T M\Lambda^{-1}, \end{aligned} \tag{3.85}$$

where $a, b, c, d \in \mathbb{R}$ and satisfy $ad - bc = 1$. The Einstein frame metric $\hat{g}$ and R-R sector four-form C_4 are $SL(2,\mathbb{R})$ invariant.

To rewrite the action in non-linearly realised form we first take a coset element $g \in SL(2,\mathbb{R})/SO(2,\mathbb{R})$ parameterised by the type IIB scalar fields ϕ and χ in the form

$$g = e^{\frac{\phi}{2}}\begin{pmatrix} e^{-\phi} & -\chi \\ 0 & 1 \end{pmatrix}. \tag{3.86}$$

The coset $g \in SL(2,\mathbb{R})/SO(2,\mathbb{R})$ is related to M by

$$\begin{aligned} gg^T =& e^{\phi}\begin{pmatrix} e^{-\phi} & -\chi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-\phi} & 0 \\ -\chi & 1 \end{pmatrix} \\ =& e^{\phi}\begin{pmatrix} |\tau|^2 & -\chi \\ -\chi & 1 \end{pmatrix} \\ =& M, \end{aligned} \tag{3.87}$$

where ^{T} denotes the transpose. One may then show

$$\begin{aligned}\left(\partial_\mu M \partial^\mu M^{-1}\right) &= -\frac{1}{4} tr\left(\left(g^{-1}\partial_\mu g + \left(g^{-1}\partial_\mu g\right)^T\right)\left(g^{-1}\partial^\mu g + \left(g^{-1}\partial^\mu g\right)^T\right)\right) \\ &= -tr\left(\mathcal{S}_\mu \mathcal{S}^\mu\right),\end{aligned} \tag{3.88}$$

where $\mathcal{S}_\mu$ are the components of the symmetric part of the $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ Cartan form $\mathcal{V} = g^{-1}dg$ constructed from $g \in SL(2,\mathbb{R})/SO(2,\mathbb{R})$. Therefore, the scalar part of the type IIB action in ten dimensions may be written in terms of the symmetric part $\mathcal{S}_\mu$ of the $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ Cartan form $\mathcal{V}_\mu$ as

$$S_{Scalar} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left(-\frac{1}{4}Tr\left(\mathcal{S}_\mu \mathcal{S}^\mu\right)\right). \tag{3.89}$$

The three form field strengths constructed from the NS-NS and R-R two-form gauge fields transform as a doublet under an $SL(2,\mathbb{R})$ transformation and appear in the type IIB effective action through a term of the form $H^a_{\mu\nu\rho}M_{ab}H^{b\mu\nu\rho}$, where a and b are two dimensional $SL(2,\mathbb{R})$ indices. Expanding the symmetric matrix M, that transforms in the adjoint representation of $SL(2,\mathbb{R})$, as $M = gg^T = g_a{}^{\overline{c}} g_b{}^{\overline{d}} \delta_{\overline{c}\overline{d}}$, where overlined indices are $SO(2,\mathbb{R})$ indices, one may convert the linear representations of $SL(2,\mathbb{R})$ carried by the three form field strengths to non-linear representations, transforming under $SO(2,\mathbb{R})$ by defining $\mathcal{H}^{\overline{c}}_{\mu\nu\rho} = g_a{}^{\overline{c}} H^a_{\mu\nu\rho}$. The three form field strengths then appear in the $d = 10$ type IIB low-energy effective action as $\mathcal{H}^{\overline{c}}_{\mu\nu\rho}\mathcal{H}^{\overline{d}\mu\nu\rho}\delta_{\overline{c}\overline{d}}$.

The scalar curvature R, five-form field strength $\tilde{F}$ and the Chern-Simons term are invariant under $SL(2,\mathbb{R})$. Therefore we may write the ten dimensional type IIB effective action at second order in derivatives in non-linearly realised form as

$$\begin{aligned}S =& \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left(R - \frac{1}{12}\mathcal{H}^{\overline{c}}_{\mu\nu\rho}\mathcal{H}^{\overline{d}\mu\nu\rho}\delta_{\overline{c}\overline{d}} + \frac{1}{4}tr\left(\mathcal{S}_\mu\mathcal{S}^\mu\right)\right) \\ &- \frac{1}{8\kappa^2}\left(\int d^{10}x\sqrt{-g}\frac{1}{5!}\tilde{F}_{\mu_1\mu_2\mu_3\mu_4\mu_5}F^{\mu_1\mu_2\mu_3\mu_4\mu_5} + \int \epsilon_{\overline{a}\overline{b}} C_4 \wedge H_3^{\overline{a}} \wedge H_3^{\overline{b}}\right).\end{aligned} \tag{3.90}$$

The $SL(2,\mathbb{R})$ indices carried by the Chern-Simons term have been converted to $SO(2,\mathbb{R})$ indices through the identity $\delta_a{}^b = g_a{}^{\overline{c}} g^b{}_{\overline{c}}$, where $g_a{}^{\overline{c}}$ and $g^b{}_{\overline{c}}$ are the components of the $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ group element g and its inverse g^{-1}, respectively. Under an $SL(2,\mathbb{R})$ transformation, all terms in the above action transform non-linearly under the $SO(2,\mathbb{R})$ maximal compact subgroup.

As mentioned earlier, in the full quantum type IIB theory the $SL(2,\mathbb{R})$ symmetry of type IIB supergravity is broken to an $SL(2,\mathbb{Z})$ S-duality subgroup. S-duality is a non-perturbative symmetry in the type IIB string coupling constant $g_{s(B)}$. In particular, an allowed transformation under the $SL(2,\mathbb{Z})$ S-duality group sends the string coupling constant $g_{s(B)}$ to its inverse $g^{-1}_{s(B)}$ and therefore relates the type IIB theory in the weak coupling regime to that in the strong coupling regime. In addition, the S-duality group $SL(2,\mathbb{Z})$ acts on the F-string that couples to the NS-NS two-form B and a D-string that couples to the R-R two-form C. An F-string in the type IIB theory carries a single unit of charge under the NS-NS two-form while the D string carries a single unit of charge under the R-R two-form. Since, the two-form gauge fields transform under the **2** of $SL(2,\mathbb{R})$, a string charged under the NS-NS two-form or the R-R two-form may transform into a (p,q) string which carries p units of NS-NS two-form charge and q units of R-R two-form charge. The restriction of the full $SL(2,\mathbb{R})$ type IIB supergravity symmetry to an $SL(2,\mathbb{Z})$ subgroup guarantees that an arbitrary type IIB string carries integer units of NS-NS and R-R two-form charge.

3.4.3 U-duality

Dimensional reduction of eleven dimensional supergravity on a circle S^1 gives type IIA supergravity with a manifest $GL(1,\mathbb{R})$ symmetry given by shifts of the type IIA dilaton with corresponding scalings of the type IIA gauge fields. Further reduction of eleven dimensional supergravity on an $n+1$ torus or type IIA on an n torus leads to a $d = 10 - n$ dimensional maximal supergravity theory that possesses a hidden $E_{n+1,n+1}(\mathbb{R})$ symmetry [8–11]. Type IIB supergravity possesses an $SL(2,\mathbb{R})$ symmetry in ten dimensions and, like type IIA supergravity, after dimensional reduction on an n torus produces a $d = 10-n$ dimensional maximal supergravity with a hidden $E_{n+1,n+1}(\mathbb{R})$ symmetry. Note that $E_{n+1,n+1}(\mathbb{R})$ is the maximally non-compact form of the exceptional Lie group E_{n+1} of rank $n+1$, with $n+1$ more non-compact generators than compact generators, however for the rest of this thesis we will simply denote this as E_{n+1}, where it is understood that we are working with the maximally non-compact form of the exceptional Lie group. The $E_{n+1}(\mathbb{Z})$ U-duality groups in each dimension d are listed in table 1.

For the full type IIA and type IIB string theories and M-theory, these hidden $E_{n+1}(\mathbb{R})$ symmetries of the massless modes found upon dimensional reduction to $d = 10 - n$ dimensions are conjectured to be broken to a $E_{n+1}(\mathbb{Z})$ U-duality group [14]. From an M-theory perspective the $E_{n+1}(\mathbb{Z})$ U-duality group can be understood to result from a non-trivial combination of the $SL(n+1,\mathbb{Z})$ subgroup that corresponds to length preserving rotations of the lattice defined by the basis elements of the torus T^{n+1} and the $O(n,n;\mathbb{Z})$ T-duality group of the type IIA theory contained in M-theory once compactified on a circle S^1. From a type IIB view point, the $E_{n+1}(\mathbb{Z})$ U-duality group combines the type IIB S-duality group $SL(2,\mathbb{Z})$ and the $O(n,n;\mathbb{Z})$ T-duality group of the type IIB theory compactified on an n torus T^n.

U-duality is a non-perturbative symmetry. Furthermore, the $O(n,n;\mathbb{Z})$ T-duality subgroup contained in the $E_{n+1}(\mathbb{Z})$ U-duality group allows one to relate the type IIB theory to the type IIA theory, and by extension, M-theory. In this sense, type IIA string theory, type IIB string theory and M-theory may be thought of as equivalent once compactified on a torus to $d < 10$ dimensions.

After dimensional reduction on an n torus T^n, the scalar fields in type IIA/B string theory and M-theory parameterise an $E_{n+1}(\mathbb{R})/H$ coset, where H is the maximal compact subgroup of the $E_{n+1}(\mathbb{R})$ U-duality group. The $d = 10 - n$ dimensional spacetime metric g transforms as an

d	$E_{n+1}(\mathbb{R})$	H	$E_{n+1}(\mathbb{Z})$
IIA	$GL(1,\mathbb{R})$	1	1
IIB	$SL(2,\mathbb{R})$	$SO(2,\mathbb{R})$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R})\times\mathbb{R}$	$SO(2,\mathbb{R})$	$SL(2,\mathbb{Z})$
8	$SL(2,\mathbb{R})\times SL(3,\mathbb{R})$	$SO(2,\mathbb{R})\times SO(3,\mathbb{R})$	$SL(2,\mathbb{Z})\times SL(3,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5,\mathbb{R})$	$SL(5,\mathbb{Z})$
6	$SO(5,5;\mathbb{R})$	$SO(5,\mathbb{R})\times SO(5,\mathbb{R})$	$SO(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8,\mathbb{R})$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8,\mathbb{R})/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$SO(16,\mathbb{R})$	$E_8(\mathbb{Z})$

Table 1: The $E_{n+1}(\mathbb{Z})$ U-duality groups and their maximal compact subgroups H

$E_{n+1}(\mathbb{Z})$ singlet, therefore the d dimensional scalar curvature R is an $E_{n+1}(\mathbb{Z})$ invariant. The derivatives of the scalars in the dimensionally reduced effective action may be written in terms of the symmetric part of the $E_{n+1}(\mathbb{R})/H$ Cartan forms $\mathcal{S}$, as explained in appendix $B.3.1$. The coupling of the gauge fields to the dimensionally reduced metric and the remaining scalar fields, either the type IIA dilaton or the type IIB dilaton and axion, results in the gauge fields transforming under linear representations of the $E_{n+1}(\mathbb{Z})$ U-duality group. In $d = 10 - n$ dimensions, the dimensionally reduced two-form field strengths lie in the representation of $E_{n+1}(\mathbb{Z})$ with highest weight Λ_1, where the E_{n+1} Dynkin diagram with the appropriate labeling is given in figure 2. Similarly, the dimensionally reduced three form field strengths constructed from the two-form

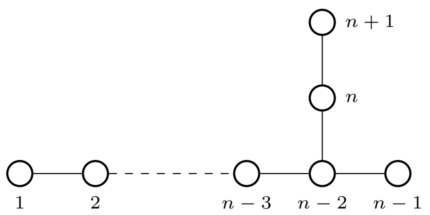

Figure 2: Dynkin diagram for E_{n+1} with type IIB labeling

gauge fields and four-form field strengths constructed from the three form gauge fields transform under linear representations of E_{n+1} with highest weights Λ_{n+1} and Λ_{n-1}, respectively. The linear representations of E_{n+1} that the d dimensional field strengths F transform under may be converted to non-linear representations $\mathcal{F}$, as described in section $B.3.2$, that transform under the maximal compact subgroup H of $E_{n+1}(\mathbb{R})$. The low-energy effective action of the dimensionally reduced theory in $d = 10 - n$ dimensions may then be constructed from the d dimensional scalar curvature R, symmetric part of the $E_{n+1}(\mathbb{R})/H$ Cartan forms $\mathcal{S}$ and the non-linear representations of the field strengths $\mathcal{F}$.

4 E_{11}

This chapter provides a brief overview of how the E_{11} Kac-Moody algebra describes the maximal supergravity theories. The techniques used to study the role of Kac-Moody algebras in supergravity and E_{11} in particular will be used extensively in chapters seven and nine.

Type IIA/B and eleven dimensional supergravity possess large and unexpected E_{n+1} hidden symmetries upon dimensional reduction on an n or $n+1$ torus, respectively. However, it has been conjectured that these symmetries are present prior to reduction and in fact are merely subgroups of a larger symmetry possessed by eleven dimensional supergravity that is described by a Kac-Moody algebra known as E_{11} [65]. In [66] it was found that eleven dimensional and type IIA supergravity could be formulated as a non-linear realisation of the Kac-Moody algebra E_{11} at low levels. The type IIB supergravity theory was similarly written as a non-linear realisation of E_{11} in reference [67]. Type IIA and type IIB supergravity may be viewed as the two different decompositions of the E_{11} algebra that give rise to an A_9 subalgebra that contains gravity and is therefore, in the context of the E_{11} Dynkin diagram, known as the gravity line. Furthermore, the maximal supergravity theories in $3 \leq d \leq 9$ dimensions, that may be identified after dimensional reduction of type IIA/B supergravity on an n torus and eleven dimensional supergravity on an $n+1$ torus, are explained through suitably decomposing the E_{11} algebra into a gravity line, A_{d-1} subalgebra, and an internal E_{n+1} subalgebra [65, 67–71].

For our purposes the E_{11} formulation of type IIA/B and eleven dimensional supergravity is used as a tool to simplify and provide insights into the group theory associated with the symmetries of these theories. The results contained in this thesis do not require the conjecture that E_{11} is a symmetry of type IIA/B and eleven dimensional supergravity to hold.

4.1 E_{11} and Kac-Moody Algebras

Kac-Moody algebras, like the classical and exceptional Lie algebras described in appendix $B.2.4$, are completely determined by the Cartan matrix defined in terms of a set of n simple roots, where n is equal to the rank of the Lie algebra. One may classify Lie algebras defined by their Cartan matrices into three different types: finite dimensional semi-simple Lie algebras have Cartan matrices that are positive definite, the Cartan matrices of infinite dimensional affine Kac-Moody algebras are positive semi-definite, while general Kac-Moody algebras, which includes E_{11}, have Cartan matrices that are indefinite. We will focus on the last case where we are dealing with a general Kac-Moody algebra, in particular E_{11}, although many of the techniques we will discuss naturally extend to the affine and finite dimensional cases.

As discussed in appendix $B.2.4$, once an $n \times n$ Cartan matrix A_{ij} is specified one may uniquely reconstruct the corresponding rank n Lie algebra by taking a Chevalley basis for the generators H_a, E_a and F_a, where H_a are the generators in the Cartan subalgebra, E_a are the positive simple root generators, F_a are the negative simple root generators and $a = 1, ..., n$. The Chevalley generators satisfy the commutation relations

$$\begin{aligned}
[H_a, H_b] &= 0, \\
[H_a, E_b] &= A_{ab} E_b, \\
[H_a, F_b] &= A_{ab} F_b, \\
[E_a, F_b] &= \delta_{ab} H_a.
\end{aligned} \tag{4.1}$$

If, in addition, we take the Chevalley generators to satisfy the further relations,

$$[E_a, [E_a, ...E_b] ...] = 0, \tag{4.2}$$

and

$$[F_a, [F_a, ...F_b] ...] = 0, \tag{4.3}$$

with $1 - A_{ab}$ E_a or F_a generators in the commutators in equations (4.2) and (4.3) respectively, then the entire Lie algebra may be constructed. The same process may be used to reconstruct a generalised Kac-Moody algebra, however, in general the algebra is infinite dimensional. As such, the adjoint representation is generally infinite dimensional so one may not define a scalar product on the generators using the trace of the adjoint representation as we can for a finite dimensional Lie algebra. Instead, it may be shown that there is a symmetric, scalar product $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on the Chevalley generators of a generalised Kac-Moody algebra $\mathfrak{g}$ defined by

$$\begin{aligned}
B(E_a, F_b) &= \delta_{ab}, \\
B(H_a, H_b) &= A_{ab},
\end{aligned} \tag{4.4}$$

where the scalar products between any other two Chevalley generators are zero. As in the finite dimensional semi-simple Lie algebra case, the roots $\vec{\alpha} \in \Delta$, defined by $[H_i, E_{\vec{\alpha}}] = \vec{\alpha}_i E_{\vec{\alpha}}$ in Cartan-Weyl basis, can be thought of as a linear functional on the Cartan subalgebra $\mathfrak{h}$ of a Kac-Moody algebra $\mathfrak{g}$. The scalar product may then be used to identify with every root $\vec{\alpha}$ a unique element of the Cartan subalgebra $h_{\vec{\alpha}} \in \mathfrak{g}$, and therefore a scalar product $(\cdot, \cdot) : \Delta \times \Delta \to \mathbb{C}$ on the roots by taking $(\vec{\alpha}, \vec{\beta}) = B(h_\alpha, h_\beta)$. The Cartan matrix of the Kac-Moody algebra may then be written in the usual form

$$A_{ab} = 2 \frac{(\vec{\alpha}_a, \vec{\alpha}_b)}{(\vec{\alpha}_a, \vec{\alpha}_a)}, \tag{4.5}$$

where $\vec{\alpha}_a$, $a = 1, ..., n$, are the simple roots of the Kac-Moody algebra.

Any Kac-Moody algebra $\mathfrak{g}$ admits a Cartan involution $\tau : \mathfrak{g} \to \mathfrak{g}$, defined in terms of the Chevalley generators by

$$\tau(E_a) = -F_a, \tag{4.6}$$

$$\tau(F_a) = -E_a, \tag{4.7}$$

$$\tau(H_a) = -H_a. \tag{4.8}$$

The Cartan involution invariant subalgebra then consists of all Kac-Moody algebra elements of the form $E_a - F_a$.

The E_{11} Dynkin diagram that summarises the corresponding Cartan matrix is given in figure 3. The eleven dimensional, type IIA and type IIB supergravity theories along with the lower dimensional maximal supergravity theories may be formulated as non-linear realisations of the semi-direct product of an E_{11} group element with another E_{11} group element with generators in the so called l_1 representation, which is the E_{11} representation with highest weight $\vec{\Lambda}_1$ using the labeling given in figure 3. In particular, the action of the maximal supergravity theories is constructed from the Cartan forms of this semi-direct product. The maximal supergravity theories are specified by selecting an A_{d-1} subalgebra, known as the gravity line, that describes the gravity sector of the d dimensional theory. To select the gravity line of a d dimensional theory we must learn how to decompose the E_{11} algebra by deleting a node or nodes of the E_{11} Dynkin diagram. After recovering the gravity line of a d dimensional maximal supergravity, the remaining E_{11} subalgebra may be interpreted as an internal algebra that generates the familiar E_{n+1} symmetry group of dimensionally reduced type IIA/B and eleven dimensional supergravity.

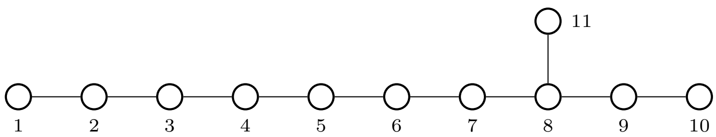

Figure 3: The E_{11} Dynkin diagram

4.2 Eleven Dimensional Supergravity

In this section we will demonstrate the basic techniques used to describe supergravity via a non-linear realisation of E_{11} by working through the eleven dimensional supergravity example.

4.2.1 Deletion of a Node

Let us first examine how one decomposes the E_{11} algebra relevant to eleven dimensional supergravity. Each simple root $\vec{\alpha}_i$, $i = 1, 2, ...11$, is associated with a positive root generator $E_{\vec{\alpha}_i}$ of the E_{11} algebra. To reproduce the required $SL(11, \mathbb{R})$ gravity line we delete node 11, as shown in figure 4. The techniques outlined in [72] may be used to show that the deletion of node 11 splits the E_{11} simple roots $\vec{\alpha}_i$, $i = 1, 2, ..., 11$, associated with node i in figure 4, into the simple roots $\vec{\alpha}_i$,

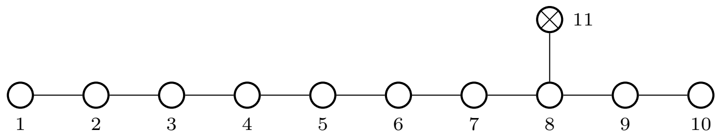

Figure 4: The E_{11} Dynkin diagram after deletion of node 11

$i = 1, 2, ..., 10$, of A_{10} given by

$$\vec{\alpha}_i = (0, \underline{\alpha}_i)\,, \quad for\ i = 1, 2, ..., 10, \tag{4.9}$$

and the simple root $\vec{\alpha}_{11}$ which may be written

$$\vec{\alpha}_{11} = (x, -\underline{\lambda}_8)\,, \tag{4.10}$$

where $\underline{\alpha}_i$ and $\underline{\lambda}_j$ are the simple roots and fundamental weights of the A_{10} subalgebra, respectively. The variable x is fixed through the requirement $(\vec{\alpha}_{11}.\vec{\alpha}_{11}) = 2$ and, since the fundamental weights of $SL(n, \mathbb{R})$ satisfy $(\lambda_i.\lambda_j) = \frac{i(n-j)}{n}$ for $i \leq j$, we have

$$x^2 = -\frac{2}{11}. \tag{4.11}$$

The corresponding E_{11} fundamental weights are then

$$\vec{\Lambda}_i = \left(\frac{\underline{\lambda}_i.\underline{\lambda}_8}{x}, \underline{\lambda}_i\right), \quad for\ i = 1, 2, ..., 10 \tag{4.12}$$

and

$$\vec{\Lambda}_{11} = \left(\frac{1}{x}, \underline{0}\right). \tag{4.13}$$

Now, since the simple roots $\vec{\alpha}_i$ provide a basis for the space of E_{11} roots, one can write any root $\vec{\alpha}$ as

$$\vec{\alpha} = \sum_{i=1}^{10} m_i \vec{\alpha}_i + l\vec{\alpha}_{11} = (lx, -\underline{\Lambda})\,, \tag{4.14}$$

where m_i and l are integers and the A_{10} part of $\vec{\alpha}$ is given by

$$\underline{\Lambda} = l\underline{\lambda}_8 - \sum_{i=1}^{10} m_i \underline{\alpha}_i. \tag{4.15}$$

The integer l is referred to as the level and counts the number of times the simple root $\vec{\alpha}_{11}$ associated with the deleted node occurs in an arbitrary E_{11} root $\vec{\alpha}$. This decomposition of the E_{11} root space allows us to evaluate the structure of the E_{11} generators $E_{\vec{\alpha}}$ corresponding to an arbitrary E_{11} simple root at each level l in terms of representations of the generators of A_{10}.

4.2.2 Representations of A_{10}

To determine the A_{10} representation content of the E_{11} generators $E_{\vec{\alpha}}$ at level l one may examine the A_{10} weight vector $\underline{\Lambda}$. Denoting the Dynkin labels of A_{10} by p_k, $k = 1, 2, ...10$, a representation of A_{10} with highest weight $\sum_k p_k \underline{\lambda}_k$ may exist at level l if there exists a level l generator $E_{\vec{\alpha}}$ with root $\vec{\alpha}$ satisfying

$$\underline{\Lambda} = \sum_k p_k \underline{\lambda}_k, \tag{4.16}$$

where $\underline{\Lambda}$ is the A_{10} weight vector defined in (4.15). Note that this is a necessary condition for an A_{10} representation to exist at level l but it is not sufficient, as an A_{10} representation at level l may have multiplicity zero and therefore not occur, however, in practice this is uncommon. One may then take the scalar product of $\underline{\Lambda}$ with $\underline{\lambda}_j$ to obtain

$$\sum_k p_k A^{-1}_{kj} = l A^{-1}_{8j} - m_j, \tag{4.17}$$

where the components of the inverse A_{10} Cartan matrix A^{-1} are the scalar products of the A_{10} fundamental weights $A^{-1}_{kj} = \left(\underline{\lambda}_k . \underline{\lambda}_j\right)$. Since the roots $\vec{\alpha}$ of any Kac-Moody algebra with a symmetric Cartan matrix are required to satisfy

$$(\vec{\alpha}, \vec{\alpha}) = 2, 0, -2, -4, ..., \tag{4.18}$$

an additional constraint on the possible A_{10} representations found at level l in the E_{11} algebra is given by the relation

$$(\vec{\alpha}, \vec{\alpha}) = -\frac{2}{11} l^2 \sum_{i,j} p_i A^{-1}_{ij} p_j = 2, 0, -2, -4, ..., \tag{4.19}$$

where we have used equation (4.14). For a given level l we then seek solutions that satisfy equations (4.17) and (4.19). A solution to these equations with non-zero Dynkin labels p_k indicates that a representation of A_{10} with highest weight $\sum_k p_k \underline{\lambda}_k$ may exist at level l.

4.2.3 Low Level Generators of the E_{11} Algebra

One may then proceed to solve equations (4.17), (4.19) for the A_{10} representations appearing at low levels. At level $l = 0$ we have the adjoint representation of A_{10}, which corresponds to the Dynkin labels $q_1 = 1$ and $q_{10} = 1$. As a result the E_{11} generators at level $l = 0$ are simply the A_{10} generators $K^a{}_b$, where $a, b = 1, 2, ..., 10$ are A_{10} indices and $\sum_a K^a{}_a = 0$. At level $l = 1$, a valid solution is $q_3 = 1$; this corresponds to an E_{11} generator R^{abc}, where a, b, c are antisymmetrised. The antisymmetry of the A_{10} indices is a generic feature of the E_{11} generators decomposed with respect to an A_n subalgebra and stems from the representations of A_n algebras with highest weights

λ_{n-k} being realised as the antisymmetric product of k vector representations. At level $l=2$ the allowed Dynkin label is $q_6=1$ which gives an E_{11} generator $R^{a_1\cdots a_6}$, where again the A_{10} indices are antisymmetrised. At level $l=3$ there are two solutions to equations (4.17) and (4.19). The first has Dynkin labels $q_1=1$ and $q_8=1$ which results in an E_{11} generator $R^{a_1\cdots a_8,b}$ where the A_{10} indices $a_1,...,a_8$ are antisymmetrised and the comma denotes that b is not antisymmetrised with any other A_{10} index. The second has the Dynkin label $q_9=1$, however the generator corresponding to this solution has multiplicity zero and therefore does not occur in the E_{11} algebra.

From the set of E_{11} generators at low levels one may identify the E_{11} Chevalley generators as [66]

$$\begin{aligned} E_a =& K^a{}_{a+1}, \quad for\ a=1,2,...,10, \\ E_{11} =& R^{9\ 10\ 11}, \end{aligned} \tag{4.20}$$

with a corresponding Cartan subalgebra

$$\begin{aligned} H_a =& K^a{}_a - K^{a+1}{}_{a+1}, \quad for\ a=1,2,...,10, \\ H_{11} =& -\frac{1}{3}\left(K^1{}_1+K^2{}_2+...+K^8{}_8\right)+\frac{2}{3}\left(K^9{}_9+K^{10}{}_{10}+K^{11}{}_{11}\right). \end{aligned} \tag{4.21}$$

The commutation relations of the low level, non-negative, E_{11} generators in the decomposition found by deleting node 11 may then be derived from the Serre relations and the Chevalley generators. The result is

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b, \tag{4.22}$$

$$[K^a{}_b, R^{c_1c_2c_3}] = \delta^{c_1}_b R^{ac_2c_3} + \delta^{c_2}_b R^{c_1ac_3} + \delta^{c_2}_b R^{c_1c_2a}, \tag{4.23}$$

$$[K^a{}_b, R^{c_1c_2\ldots c_6}] = \delta^{c_1}_b R^{ac_2\ldots c_6} + \delta^{c_2}_b R^{c_1ac_3\ldots c_6} + ... + \delta^{c_2}_b R^{c_1c_2\ldots c_5a}, \tag{4.24}$$

$$\left[K^a{}_b, R^{c_1c_2\ldots c_8,d}\right] = \delta^{c_1}_b R^{ac_2\ldots c_8,d} + \delta^{c_2}_b R^{c_1ac_3\ldots c_8,d} + ... + \delta^{c_2}_b R^{c_1c_2\ldots c_7a,d} + \delta^d_b R^{c_1c_2\ldots c_8,a}, \tag{4.25}$$

$$[R^{c_1c_2c_3}, R^{c_4c_5c_6}] = 2R^{c_1c_2\ldots c_6}, \tag{4.26}$$

$$[R^{a_1a_2\ldots a_6}, R^{c_1c_2c_3}] = 3R^{a_1a_2\ldots a_6[c_1c_2c_3]}. \tag{4.27}$$

The remaining infinite set of positive root generators are, in principal, found from taking the multiple commutators of the positive root generators with R^{abc} and imposing the Chevalley-Serre relations (4.2).

4.2.4 l_1 Representation

The l_1 representation of E_{11} is the representation of the E_{11} algebra with highest weight $\vec{\Lambda}_1$ and appears in a semi-direct product with the E_{11} group element parameterised by the generators decomposed with respect to the A_{10} subalgebra derived in the previous section [73]. In general this semi-direct product is specified by taking a Lie algebra $\mathfrak{g}$ and a particular representation of the Lie algebra such that when acting on the states $|X_i\rangle$ in the representation space one has

$$U(A)|X_i\rangle = -D(A)_i{}^j|\chi_j\rangle \tag{4.28}$$

where $A \in \mathfrak{g}$, U is a representation of $\mathfrak{g}$, $D(A)_i{}^j$ are the components of the representation U in matrix form and $|\chi_j\rangle$ are the states of the representation in vector form. For each state in the representation $|X_i\rangle$ one may define an element X_i in an enlarged algebra by taking

$$[X_i, A] = -D\,(A)_i{}^j X_j. \tag{4.29}$$

The Jacobi identities for the enlarged algebra are satisfied provided that the commutation relations for the elements X_i are derived from the chosen representation. We will take the case where the Lie algebra is the E_{11} algebra decomposed with respect to the A_{10} subalgebra and the enlarged algebra contains the generators derived from the l_1 representation along with the E_{11} generators. To ensure that this is indeed a Lie algebra one must specify a set of commutation relations that lead to a set of Jacobi identities for the enlarged algebra. To do so we will first examine the generators in the enlarged algebra derived from the l_1 representation.

To calculate the weights of the l_1 representation and therefore deduce the E_{11} generators in the l_1 representation in terms of the A_{10} subalgebra one may use a similar technique to that outlined in the previous section. Except that in this case before deleting node 11 we append the E_{11} Dynkin diagram by a node labelled $\star$ with one edge joining node 1 and the added node before deleting both the added node and node 11, as demonstrated in figure 5. The reader is referred to reference [74] for

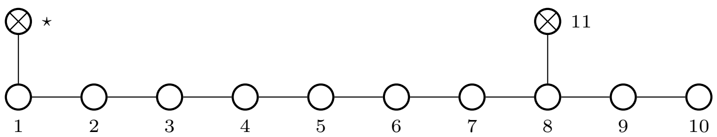

Figure 5: The E_{11} Dynkin diagram after deletion of the added node and node 11

details of the derivation of the l_1 representation. The resulting generators in the l_1 representation at low levels, in terms of the A_{10} subalgebra, are P_a at level $l = 0$, Z^{ab} at level $l = 1$ and $Z^{a_1 \dots a_5}$ at level $l = 2$, where P_a is identified as the translation generator, Z^{ab} the two brane charge and $Z^{a_1 \dots a_5}$ the five brane charge. The remaining infinite set of generators in the l_1 representation may be interpreted as the charges of the branes that couple to the infinite set of fields that appear as the parameters of the E_{11} generators decomposed with respect to the A_{10} subalgebra [73, 75–77]. The parameters of the l_1 generators P_a, Z^{ab} and $Z^{a_1 \dots a_5}$ are taken to be x^a, x_{ab} and $x_{a_1 \dots a_5}$, respectively, where the parameters x^a are the coordinates of the background spacetime and the parameters of the generators at higher levels are coordinates associated with the brane charges.

The commutation relations for the generators in the original E_{11} algebra with the generators derived from the l_1 representation of the E_{11} algebra at levels $l = 0$ and $l = 1$ are

$$[K^a{}_b, P_c] = -\delta^a_c P_b + \frac{1}{2}\delta^a_b P_c, \tag{4.30}$$

$$[K^a{}_b, Z^{c_1c_2}] = 2\delta^{[c_1}_b Z^{|a|c_2]} + \frac{1}{2}\delta^a_b Z^{c_1c_2}, \tag{4.31}$$

$$[K^a{}_b, Z^{c_1c_2\ldots c_5}] = 5\delta^{[c_1}_b Z^{|a|c_2\ldots c_5]} + \frac{1}{2}\delta^a_b Z^{c_1c_2\ldots c_5}, \tag{4.32}$$

$$[R^{a_1a_2a_3}, P_b] = 3\delta^{[a_1}_b Z^{a_2a_3]}, \tag{4.33}$$

$$\left[R^{a_1a_2a_3}, Z^{b_1b_2}\right] = Z^{a_1a_2a_3b_1b_2}, \tag{4.34}$$

additional commutation relations for higher level generators are given in references [73, 78].

4.2.5 E_{11} Cartan Forms

The appropriate E_{11} Cartan form that reproduces the eleven dimensional supergravity equations of motion is constructed from a group element g that is the semi-direct product of the E_{11} group element containing the eleven dimensional supergravity fields as the parameters of the generators decomposed with respect to the A_{10} subalgebra and another E_{11} group element containing the space-time coordinates as the parameters of the translation generators P_a in the l_1 representation. In general the group element g is given by a product of exponentials of the full, infinite set of generators in both the l_1 representation and the adjoint representation of E_{11}. However, in this thesis we will truncate the enlarged E_{11} algebra such that all generators in the l_1 representation at levels $l > 0$ are set to zero along with all generators in the adjoint representation at levels $l > 2$, this subalgebra is known in the literature as G_{11}. One may then take the E_{11} group element g to be of the form

$$g = e^{x^\mu P_\mu} e^{h_a{}^b K^a{}_b} e^{\frac{1}{3!}A_{c_1c_2c_3}R^{c_1c_2c_3}} e^{\frac{1}{6!}A_{d_1d_2\ldots d_6}R^{d_1d_2\ldots d_6}}. \tag{4.35}$$

The non-linear realisation is constructed by considering group elements g that belong to the semi-direct product of the l_1 representation of E_{11} with the adjoint representation of E_{11} and transform as

$$g \to g_0 g h^{-1}, \tag{4.36}$$

where g_0 is a rigid E_{11} group element and h is a member of the local group generated by the Cartan involution invariant subalgebra. The reader is referred to appendix $B.3$ for further details on non-linear realisations. Defining the one-form $\mathcal{V}$ by

$$\mathcal{V} = g^{-1}dg - \omega, \tag{4.37}$$

where $\omega = \frac{1}{2} dx^{\mu} \omega_{\mu b}{}^{c} J^{b}{}_{c}$ is the Lorentz connection that transforms as

$$\omega \rightarrow h \omega h^{-1} + h d h^{-1}. \tag{4.38}$$

It follows from equations (4.36) and (4.38) that $\mathcal{V}$ transforms purely under the group generated by the Cartan involution invariant subalgebra

$$\mathcal{V} \rightarrow h \mathcal{V} h^{-1}. \tag{4.39}$$

Calculating V for the parameterisation of the group element given in equation (4.35) one finds

$$V = dx^{\mu} \left(e_{\mu}{}^{a} P_{a} + \Omega_{\mu a}{}^{b} K^{a}{}_{b} + \frac{1}{3!} \tilde{D}_{\mu} A_{c_1 c_2 c_3} R^{c_1 c_2 c_3} + \frac{1}{6!} \tilde{D}_{\mu} A_{c_1 c_2 \ldots c_6} R^{c_1 c_2 \ldots c_6} \right), \tag{4.40}$$

where

$$e_{\mu}{}^{a} = \left(e^{h}\right)_{\mu}{}^{a}, \tag{4.41}$$

$$\tilde{D}_{\mu} A_{c_1 c_2 c_3} = \partial_{\mu} A_{c_1 c_2 c_3} + \left(e^{-1} \partial_{\mu} e\right)_{c_1}{}^{b} A_{b c_2 c_3} + \left(e^{-1} \partial_{\mu} e\right)_{c_2}{}^{b} A_{c_1 b c_3} + \left(e^{-1} \partial_{\mu} e\right)_{c_3}{}^{b} A_{c_1 c_2 b}, \tag{4.42}$$

$$\begin{aligned} \tilde{D}_{\mu} A_{c_1 c_2 \ldots c_6} = {} & \partial_{\mu} A_{c_1 c_2 \ldots c_6} + \left(e^{-1} \partial_{\mu} e\right)_{c_1}{}^{b} A_{b c_2 \ldots c_6} \\ & + \left(e^{-1} \partial_{\mu} e\right)_{c_2}{}^{b} A_{c_1 b c_3 \ldots c_6} + \ldots + \left(e^{-1} \partial_{\mu} e\right)_{c_6}{}^{b} A_{c_1 c_2 \ldots c_5 b}, \end{aligned} \tag{4.43}$$

$$\Omega_{\mu a}{}^{b} = \left(e^{-1} \partial_{\mu} e\right)_{a}{}^{b} - \omega_{\mu a}{}^{b}. \tag{4.44}$$

One may also calculate the Cartan forms of the eleven dimensional conformal group which is contained in E_{11}. Preserving the Cartan forms of (4.40) that may be rewritten in terms of the Cartan forms of the eleven dimensional conformal group, such that the resulting Cartan forms are simultaneously covariant under the eleven dimensional conformal group and the group generated by G_{11}, yields a set of first order equations that are equivalent to the equations of motion of eleven dimensional supergravity. The reader is referred to references [65, 66, 79] for details.

It is thus possible to describe eleven dimensional supergravity as a non-linear realisation of an E_{11} Kac-Moody algebra. Moreover, the fields are in one-to-one correspondence with the E_{11} generators.

4.3 Type IIA Supergravity

The decomposition of the E_{11} algebra that explicitly describes type IIA supergravity is given by deleting nodes 10 and 11 in the E_{11} Dynkin diagram, as shown in figure 6. A derivation of the simple roots and fundamental weights relative to the A_9 subalgebra found by deleting nodes 10 and 11 is given in appendix $D.5$. The low level generators of E_{11} decomposed with respect to the A_9 subalgebra relevant to type IIA supergravity are $K_{a}{}^{b}$ and the set of generators $R^{a_1 a_2 \ldots a_m}$ for

$m = 1, ..., 9$ and the A_9 indices a, b, $a_1, ..., a_9$, range from 1 to 10.

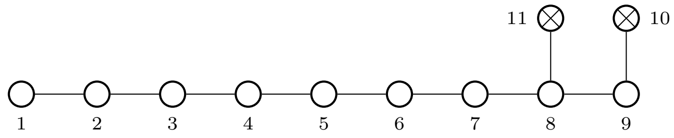

Figure 6: The E_{11} Dynkin diagram appropriate to type IIA supergravity

The E_{11} Chevalley generators with respect to the type IIA supergravity decomposition are [66]

$$E_a = K^a{}_{a+1}, \quad for\ a = 1, 2, ..., 9, \tag{4.45}$$

$$E_{10} = R^{10}, \tag{4.46}$$

$$E_{11} = R^{9\ 10}, \tag{4.47}$$

while the corresponding Cartan subalgebra is given by

$$\begin{aligned} H_a &= K^a{}_a - K^{a+1}{}_{a+1}, \quad a = 1, ..., 9, \\ H_{10} &= -\frac{1}{8}\left(K^1{}_1 + ... + K^9{}_9\right) + \frac{7}{8}K^{10}{}_{10} - \frac{3}{2}R, \\ H_{11} &= -\frac{1}{4}\left(K^1{}_1 + ... + K^8{}_8\right) + \frac{3}{4}\left(K^9{}_9 + K^{10}{}_{10}\right) + R. \end{aligned} \tag{4.48}$$

The l_1 representation for the decomposition of E_{11} with respect to the A_9 subalgebra of the type IIA theory contains the translation generators P_a at level $l = 0$ along with an infinite set of generators at higher levels that correspond to the type IIA brane charges. The commutation relations for the low level generators in addition to the translation generators P_a of the l_1 representation are

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^b{}_c, \tag{4.49}$$

$$[K^a{}_b, P_c] = -\delta^a_c P_b, \tag{4.50}$$

$$[K^a{}_b, R^{c_1 c_2 \ldots c_m}] = \delta^{c_1}_b R^{a c_2 \ldots c_m} + \delta^{c_2}_b R^{c_1 a \ldots c_m} + ... + \delta^{c_m}_b R^{c_1 c_2 \ldots a}, \tag{4.51}$$

$$[R, R^{c_1 c_2 \ldots c_m}] = c_m R^{c_1 c_2 \ldots c_m}, \tag{4.52}$$

$$[R^{c_1 c_2 \ldots c_m}, R^{c_1 c_2 \ldots c_p}] = c_{m,p} R^{c_1 c_2 \ldots c_{m+p}}, \tag{4.53}$$

where the constants c_p and $c_{m,p}$ are given in reference [66].

Type IIA supergravity may then be formulated by taking a non-linear realisation of the E_{11} algebra decomposed with respect to the A_9 subalgebra found by deleting nodes 10 and 11. The group element g_{IIA} constructed out of the semi-direct product of the l_1 representation at level zero and the lower level E_{11} generators is given by

$$g_{IIA} = e^{x^a P_a} e^{h_a{}^b K^a{}_b} e^{AR} e^{A_a R^a} e^{\frac{1}{2!} A_{a_1 a_2} R^{a_1 a_2}} ... e^{\frac{1}{9!} A_{a_1 a_2 ... a_9} R^{a_1 a_2 ... a_9}}. \tag{4.54}$$

The fields that appear as the parameters of the corresponding generators in the E_{11} group element g_{IIA} are the fields of type IIA supergravity, where $h_a{}^b$ is the graviton, A is the type IIA dilaton and $A_{a_1 a_2 ... a_m}$ are the m form gauge fields.

4.4 Type IIB Supergravity

The other decomposition of the E_{11} algebra, found by deleting node 10 as shown in figure 7, leads to an $SL(10)$ gravity line that describes the IIB theory. One immediately sees that, in addition to yielding an A_9 subalgebra, the deletion of node 10 in figure 7 gives an A_1 subalgebra that corresponds to the expected $SL(2, \mathbb{R})$ symmetry of type IIB supergravity. A derivation of the simple roots and fundamental weights relative to the A_9 subalgebra found by deleting node 10 is given in appendix $D.3$. The low level generators of E_{11} decomposed with respect to the A_9 and A_1 subalgebras relevant to type IIB supergravity are $K_a{}^b$ and the set of generators R_s, $R_s^{a_1 a_2}$, $R_2^{a_1 a_2 a_3 a_4}$, $R_s^{a_1 a_2 ... a_6}$, $R_s^{a_1 a_2 ... a_8}$, where $s = 1, 2$ is an $SL(2)$ index and the A_9 indices a, b, a_1,... a_8, range from 1 to 10.

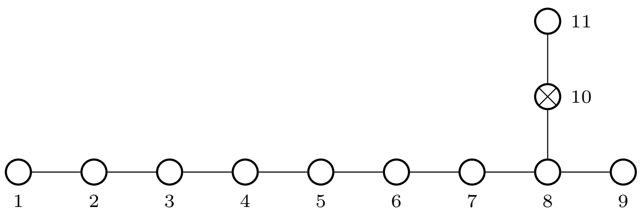

Figure 7: The E_{11} Dynkin diagram appropriate to type IIB supergravity

The E_{11} Chevalley generators with respect to the type IIB supergravity decomposition are [67]

$$E_a = K^a{}_{a+1}, \quad for\ a = 1, 2, ..., 8, \tag{4.55}$$

$$E_9 = R_1^{9\ 10}, \tag{4.56}$$

$$E_{10} = R_2, \tag{4.57}$$

$$E_{11} = K^9{}_{10}, \tag{4.58}$$

while the corresponding Cartan subalgebra is given by

$$\begin{aligned} H_a &= K^a{}_a - K^{a+1}{}_{a+1}, \quad for\ a = 1, ..., 8, \\ H_9 &= K^9{}_9 + K^{10}{}_{10} + R_1 - \frac{1}{4}\sum_{a=1}^{10} K^a{}_a, \\ H_{10} &= -2R_1, \\ H_{11} &= K^9{}_9 - K^{10}{}_{10}. \end{aligned} \tag{4.59}$$

As usual the l_1 representation for the decomposition of E_{11} with respect to the A_9 subalgebra of the type IIB theory contains the translation generators P_a at level $l = 0$ along with an infinite set of generators at higher levels that are identified as the type IIB brane charges. The commutation relations for the low level generators in addition to the translation generators P_a of the l_1 representation are

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^b{}_c, \tag{4.60}$$

$$[K^a{}_b, P_c] = -\delta^a_c P_b + \frac{1}{2}\delta^a_b P_c, \tag{4.61}$$

$$[K^a{}_b, R^{c_1 c_2 \ldots c_m}_s] = \delta^{c_1}_b R^{b c_2 \ldots c_m}_s + \delta^{c_2}_b R^{c_1 b \ldots c_m}_s + \ldots + \delta^{c_m}_b R^{c_1 c_2 \ldots b}_s, \tag{4.62}$$

$$[R_1, R^{c_1 c_2 \ldots c_m}] = d^s_m R^{c_1 c_2 \ldots c_m}_s, \tag{4.63}$$

$$[R_2, R^{c_1 c_2 \ldots c_m}] = \tilde{d}^s_m R^{c_1 c_2 \ldots c_m}_s, \tag{4.64}$$

$$\left[R^{c_1 c_2 \ldots c_m}_{s_1}, R^{c_1 c_2 \ldots c_p}_{s_2}\right] = c^{s_1, s_2}_{m,p} R^{c_1 c_2 \ldots c_{m+p}}, \tag{4.65}$$

where the constants d^s_m, $\tilde{d}^s_m$ and $c^{s_1,s_2}_{m,p}$ are given in reference [67].

Type IIB supergravity may then be formulated by taking a non-linear realisation of the E_{11} algebra decomposed with respect to the A_9 subalgebra found by deleting node 10. The group element g_{IIB} constructed out of the semi-direct product of the l_1 representation at level zero and the lower level E_{11} generators is given by

$$\begin{aligned} g_{IIB} = {} & e^{x^a P_a} e^{h_a{}^b K^a{}_b} e^{A^1 R_1} e^{A^2 R_2} e^{\frac{1}{2!} A^s_{a_1 a_2} R^{a_1 a_2}_s} e^{\frac{1}{4!} A^2_{a_1 a_2 a_3 a_4} R^{a_1 a_2 a_3 a_4}_2} e^{\frac{1}{6!} A^s_{a_1 a_2 \ldots a_6} R^{a_1 a_2 \ldots a_6}_s} \\ & \times e^{\frac{1}{8!} A^s_{a_1 a_2 \ldots a_8} R^{a_1 a_2 \ldots a_8}_s}. \end{aligned} \tag{4.66}$$

The fields that appear as the parameters of the corresponding generators in the E_{11} group element g_{IIB} are the fields of type IIB supergravity, where $h_a{}^b$ is the graviton, A^1 is the type IIB dilaton, A^2 is the type IIB axion and $A^s_{a_1 a_2 \ldots a_m}$ are the m form gauge fields.

4.5 Maximal Supergravity in $d < 10$ Dimensions

The story is similar for the maximal supergravity theories in $d < 10$ dimensions arising from dimensional reduction of type IIA/B or eleven dimensional supergravity. A d dimensional supergravity theory requires an $SL(d)$ gravity line, therefore one deletes node d in the E_{11} Dynkin diagram, as shown in figure 8 and in addition we recover the E_{n+1} subalgebra of the $d = 10 - n$ dimensional U-duality group. One may then find the E_{11} algebra decomposed with respect to the A_{d-1}

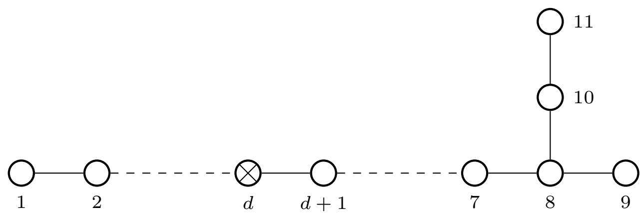

Figure 8: The E_{11} Dynkin diagram appropriate to maximal supergravity in $d < 10$ dimensions

subalgebra and construct a non-linear realisation from a group element generated by the d dimensional decomposition of the E_{11} algebra that describes the d dimensional maximal supergravity theory. The details of constructing the d dimensional maximal supergravity theories as non-linear realisations of E_{11} are found in references [65, 67–71]. In chapters seven and nine we will use the $d = 10 - n$ dimensional decomposition of the E_{11} algebra to identify the physical fields, which we define as the set of scalars found upon dimensional reduction of the type IIA/B and eleven dimensional supergravity theories, in terms of the E_{n+1} fields that act as the parameters of the Cartan subalgebra in the E_{n+1} group element.

5 The Type IIB String Theory effective action in $d = 7$ Dimensions

In this chapter we will examine the dimensional reduction of the effective action of type IIB string theory on a three torus to $d = 7$ dimensions. Type IIB string theory compactified on a three torus provides a relatively straightforward setting to begin our investigation into the higher order terms in the effective action of type IIB string theory and their automorphic forms in $d < 10$ dimensions. In particular, the well known R^4 term in the effective action of type IIB string theory on a three torus possesses a coefficient function that is an $SL(5,\mathbb{Z})$ Eisenstein series with a simple construction. In lower dimensions the construction of the coefficient function of the R^4 term is complicated by required constraints, this is discussed in chapter eight.

The classical type IIB supergravity theory dimensionally reduced on a three torus to $d = 7$ dimensions possesses a hidden $SL(5,\mathbb{R})$ symmetry. It is conjectured that an $SL(5,\mathbb{Z})$ subgroup of $SL(5,\mathbb{R})$ is a symmetry of the full quantum type IIB string theory after compactification on a three torus. After dimensional reduction of the two derivative terms in the effective action of type IIB string theory to $d = 7$ dimensions we proceed to identify the building blocks of the effective action in non-linearly realised form. Higher derivative terms in the effective action are then constructed from $SL(5,\mathbb{R})/SO(5,\mathbb{R})$ Cartan forms, non-linearly realised field strengths, and seven dimensional Riemann curvatures and possess an $SL(5,\mathbb{Z})$ automorphic form as a coefficient function.

The mixing of the ten dimensional type IIB fields in the multiplets of the the non-linearly realised $SL(5,\mathbb{Z})$ formulation of the type IIB action in seven dimensions allows us to conjecture that given a higher derivative term consisting of a polynomial in the ten dimensional fields, containing l derivatives, with a coefficient function that transforms as an $SL(2,\mathbb{Z})$ automorphic form $\Phi_{SL(2)}$ in the ten dimensional type IIB effective action one should find a set of higher derivative terms in the ten dimensional IIB effective action consisting of related but different polynomials in the ten dimensional fields, containing l derivatives, carrying the same $SL(2,\mathbb{Z})$ automorphic form $\Phi_{SL(2)}$.

For instance, the type IIB R^4 term in ten dimensions has a coefficient function which is an $SL(2,\mathbb{Z})$ Eisenstein series $\Phi_{SL(2)}$ with $s = \frac{3}{2}$ [15]. From our conjecture one should find a set of higher derivative terms in the ten dimensional type IIB low-energy effective that are eight derivative polynomials in the ten dimensional Riemann curvature R, $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ Cartan forms, non-linearly realised three form field strengths and the five-form field strength that carry the same $SL(2,\mathbb{Z})$ automorphic form $\Phi_{SL(2)}$ with $s = \frac{3}{2}$.

In addition, we find that the building blocks of the non-linearly realised type IIB effective action in $d = 7$ dimensions carry a factor of $e^{-\alpha\rho}$, where ρ is the volume modulus of the three torus, for each derivative in a given term. This observation is followed up in chapter 6 and leads to constraints on the automorphic forms of the higher derivative terms in $d < 10$ dimensions.

5.1 Dimensionally Reduced $d = 7$ Action

The $SL(5,\mathbb{Z})$ symmetry of the two derivative terms in the type IIB string theory effective action on a three torus is not immediately manifest after dimensional reduction. To write the dimensionally reduced type IIB effective action in $SL(5,\mathbb{Z})$ invariant form we will make use of the theory of non-linear realisations described in appendix $B.3$.

In this formulation, the derivatives of the scalars, which parameterise an $SL(5,\mathbb{R})/SO(5,\mathbb{R})$ coset $E \in SL(5,\mathbb{R})/SO(5,\mathbb{R})$ appear in the effective action through the symmetric part (under the Cartan involution) of the $SL(5,\mathbb{R})/SO(5,\mathbb{R})$ Cartan forms, constructed from the coset E. The seven dimensional curvature R transforms trivially since the seven dimensional metric is an $SL(5,\mathbb{Z})$ singlet. The two and three form field strengths F constructed from the dimensionally reduced gauge fields which transform under linear representations of $SL(5,\mathbb{Z})$ may be converted to non-linear representations $\mathcal{F}$ of $SL(5,\mathbb{Z})$ transforming under $SO(5,\mathbb{R})$ through the use of the coset $E \in SL(5,\mathbb{R})/SO(5,\mathbb{R})$, as summarised in appendix ($B.3.2$). We are also left with four and five-form field strengths after dimensional reduction but these originate from dimensional reduction of the self dual five-form field strength in ten dimensions and are therefore related to the two and three form field strengths in seven dimensions and do not provide independent degrees of freedom. The familiar, lowest order, terms in the effective action may then be written purely in terms of the seven dimensional curvature R, the symmetric part of the $SL(5,\mathbb{R})/SO(5,\mathbb{R})$ Cartan forms $\mathcal{S}$, and the non-linear representations of the two and three form field strengths $\mathcal{F}$. We will discuss the higher order terms in the $d = 7$ type IIB effective action after formulating the lowest order terms in this way.

As shown in section 3.4.2, the type IIB effective action at second order in derivatives may be written in non-linearly realised form with respect to the $SL(2,\mathbb{Z})$ symmetry of the $d = 10$ type IIB theory. After expanding out the $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ Cartan forms in (3.90) the type IIB effective action at second order in derivatives is

$$S_{IIB} = \frac{1}{2\kappa_{10}^2}\int\left(\hat{R} - \frac{1}{2}d\phi \wedge *d\phi - \frac{1}{2}d\chi \wedge *d\chi e^{2\phi} - \frac{1}{2}\delta_{\overline{a}\overline{b}}\hat{H}^{(3)\overline{a}} \wedge *\hat{H}^{(3)\overline{b}} - \frac{1}{4}\hat{F}^{(5)} \wedge *\hat{F}^{(5)}\right), \tag{5.1}$$

where overlined indices $\overline{a}, \overline{b}$ are local $SO(2,\mathbb{R})$ indices, indicating the three form field strengths $\hat{H}^{(3)\overline{a}}$ transform non-linearly under the $SL(2,\mathbb{Z})$ symmetry. Note that the $d = 10$ type IIB Chern-Simons term has been suppressed for now. Earlier we mentioned that the $d = 10$ type IIB effective action must be supplemented with a self duality constraint on the five-form field strength $\hat{F}^{(5)}$ after calculating the equations of motion. However, it was demonstrated in [80, 81] that one could consistently implement the self duality constraint on the five-form field strength after dimensional reduction, this gives a set of equations relating the components of the dimensionally reduced five-form field strength. In odd dimensions it is then possible to eliminate half the degrees of freedom associated with the five-form field strength by substituting for half the dimensionally reduced components of the five-form field strength in terms of the remaining components in the dimensionally reduced action. In this way we are able to write down a valid $d = 7$ dimensionally reduced type IIB action.

We proceed to compactify three of the dimensions through the methods described in chapter 3. We will take a compactification ansatz

$$d\hat{s}^2 = e^{2\alpha\rho}ds^2 + e^{2\beta\rho}G_{ij}\left(dx^i + A^i_\mu dx^\mu\right)\left(dx^j + A^j_\mu dx^\mu\right), \tag{5.2}$$

where a $\hat{}$ denotes a $D = 10$ quantity, while all fields without a $\hat{}$ are lower dimensional $D = 7$ fields and, as usual, the metric on the torus G satisfies $det\,(G) = 1$. Our choice of vielbein frame for this compactification is

$$\begin{aligned}\hat{e}^{\overline{\nu}} =& e^{\alpha\rho}e_\mu{}^{\overline{\nu}}dx^\mu,\\ \hat{e}^{\overline{i}} =& e^{\beta\rho}e_j{}^{\overline{i}}\left(dx^j + A^j_\mu dx^\mu\right),\end{aligned} \tag{5.3}$$

where $e_\mu{}^{\overline{\nu}}$ is a vielbein for the background $D = 7$ metric and $e_j{}^{\overline{i}}$ is a vielbein for the metric on the torus G. The vielbein on the torus $e \in SL(3)/SO(3)$ will give the action for the scalar fields, arising through expressing the $D = 10$ scalar curvature in terms of $D = 7$ fields, as a trace over the symmetric part of its Cartan form $e_{\overline{j}}{}^k\partial_\mu e_k{}^{\overline{i}}$, for which we will take the split

$$e_{\overline{j}}{}^k\partial_\mu e_k{}^{\overline{i}} = S_{\overline{\mu}}{}^{\overline{i}}{}_{\overline{j}} + Q_{\overline{\mu}}{}^{\overline{i}}{}_{\overline{j}}, \tag{5.4}$$

and $Q_\mu^{[\overline{i}\overline{j}]} = S_\mu^{(\overline{i}\overline{j})} = 0$, Q and S denoting the antisymmetric and symmetric parts respectively of the Cartan form. Our choice of vielbein frame yields a spin connection

$$\begin{aligned}\hat{\omega}^{\overline{i}}{}_{\overline{j}} =& -e^{-\alpha\rho}Q_{\overline{\mu}}{}^{\overline{i}}{}_{\overline{j}}\hat{e}^{\overline{\mu}},\\ \hat{\omega}^{\overline{i}}{}_{\overline{\nu}} =& \beta\left(\partial_{\overline{\nu}}\rho\right)e^{-\alpha\rho}\hat{e}^{\overline{i}} + e^{-\alpha\rho}\left(S_{\overline{\nu}}{}^{\overline{i}}{}_{\overline{j}} + Q_{\overline{\nu}}{}^{\overline{i}}{}_{\overline{j}}\right)\hat{e}^{\overline{j}} + \frac{1}{2}e^{(\beta-2\alpha)\rho}F^{\overline{i}}_{\overline{\mu}\overline{\nu}},\\ \hat{\omega}^{\overline{\mu}}{}_{\overline{\nu}} =& \omega^{\overline{\mu}}{}_{\overline{\nu}} + \alpha\partial_{\overline{\nu}}\rho e^{-\alpha\rho}\hat{e}^{\overline{\mu}} - \alpha\partial^{\overline{\mu}}\rho e^{-\alpha\rho}\hat{e}_{\overline{\nu}} - \frac{1}{2}e^{(\beta-2\alpha\rho)\rho}F_{\overline{i}}{}^{\overline{\mu}}{}_{\overline{\nu}},\end{aligned} \tag{5.5}$$

and scalar curvature

$$\hat{R} = e^{-2\alpha\rho}\left(R - \frac{1}{4}e^{2(\beta-\alpha)\rho}G_{ij}\tilde{F}^i_{\mu\nu}\tilde{F}^{j\mu\nu} - S_{\mu i}{}^jS^{\mu i}_j - \gamma^2\left(\partial\rho\right)^2 - 2\left(n\beta + (d-1)\,\alpha\right)\nabla^2\rho\right), \tag{5.6}$$

where $\tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\gamma^2 = \frac{1}{2}$ for dimensional reduction to Einstein frame, and for $D = 10$, $d = 7$ and $n = 3$ we have $\beta = -\frac{5}{3}\alpha$ and $\alpha^2 = \frac{3}{80}$ from equations (3.12) and (3.13).

Dimensional reduction of the ten dimensional two-form gauge fields $\hat{B}^{(2)\overline{a}}$ gives

$$\hat{B}^{\overline{a}} = B^{(2)\overline{a}} + B^{(1)\overline{a}}_{i_1} \wedge dx^{i_2} + \frac{1}{2!}B^{(0)\overline{a}}_{i_1i_2} \wedge dx^{i_1} \wedge dx^{i_2}. \tag{5.7}$$

Therefore, dimensional reduction of the ten dimensional three form field strengths gives

$$\begin{aligned}\hat{H}^{\overline{a}} &= \frac{1}{3!} H^{(3)\overline{a}}_{\mu_1\mu_2\mu_3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \\ &+ \frac{1}{2!} H^{(2)\overline{a}}_{\mu_1\mu_2 i_1} dx^{\mu^1} \wedge dx^{\mu_2} \wedge dx^{i_1} \\ &+ \frac{1}{2!} H^{(1)\overline{a}}_{\mu_1 i_1 i_2} dx^{\mu^1} \wedge dx^{i_1} \wedge dx^{i_2}.\end{aligned} \tag{5.8}$$

Converting to tangent frame through the relations

$$dx^{\mu} = e^{-\alpha\rho} e_{\overline{\nu}}{}^{\mu} \hat{e}^{\overline{\nu}} \tag{5.9}$$

and

$$dx^{i} = e^{-\beta\rho} e_{\overline{j}}{}^{i} \hat{e}^{\overline{j}} - e^{-\alpha\rho} A^{i}_{\mu} e_{\overline{\nu}}{}^{\mu} \hat{e}^{\nu}, \tag{5.10}$$

one finds

$$\begin{aligned}\hat{H}^{\overline{a}} = \frac{1}{3!} e^{-3\alpha\rho} H^{(3)\overline{a}}_{\overline{\mu}_1\overline{\mu}_2\overline{\mu}_3} \hat{e}^{\overline{\mu}_1} \wedge \hat{e}^{\overline{\mu}_2} \wedge \hat{e}^{\overline{\mu}_3} + \frac{1}{3!} H^{(2)\overline{a}}_{\mu_1\mu_2 i_1} \left(e^{-2\alpha\rho} \hat{e}^{\overline{\mu}_1} \wedge \hat{e}^{\overline{\mu}_2}\right) \wedge \left(e^{-\beta\rho} e_{\overline{i}_1}{}^{i_1} \hat{e}^{\overline{i}_1} - e^{-\alpha\rho} A^{i_1}_{\overline{\mu}_3} \hat{e}^{\overline{\mu}_3}\right) \\ + \frac{1}{2!} H^{(1)\overline{a}}_{\mu_1 i_1 i_2} \left(e^{-\alpha\rho} \hat{e}^{\overline{\mu}_1} \wedge \left(e^{-\beta\rho} e_{\overline{i}_1}{}^{i_1} \hat{e}^{\overline{i}_1} - e^{-\alpha\rho} A^{i_1}_{\overline{\mu}_2} \hat{e}^{\overline{\mu}_2}\right) \wedge \left(e^{-\beta\rho} e_{\overline{i}_2}{}^{i_2} \hat{e}^{\overline{i}_2} - e^{-\alpha\rho} A^{i_2}_{\overline{\mu}_3} \hat{e}^{\overline{\mu}_3}\right)\right).\end{aligned} \tag{5.11}$$

As mentioned in chapter 3, where we describe the dimensional reduction process, our convention will be to shuffle all internal space basis elements to the right of all spacetime basis elements and shuffle all contractions between the graviphoton internal indices and field strength internal indices to the right of all free indices by making use of the antisymmetry of the field strength indices. Employing these conventions in (5.11), gives

$$\begin{aligned}\hat{H}^{\overline{a}} &= e^{-3\alpha\rho} \left(\frac{1}{3!} H^{(3)\overline{a}}_{\overline{\mu}_1\overline{\mu}_2\overline{\mu}_3} - \frac{1}{2!} H^{(2)\overline{a}}_{\overline{\mu}_1\overline{\mu}_2\overline{i}_1} \tilde{A}^{\overline{i}_1}_{\overline{\mu}_3} + \frac{1}{2!} H^{(1)\overline{a}}_{\overline{\mu}_1\overline{i}_1\overline{i}_2} A^{\overline{i}_1}_{\overline{\mu}_2} \tilde{A}^{\overline{i}_2}_{\overline{\mu}_3}\right) \hat{e}^{\overline{\mu}_1} \wedge \hat{e}^{\overline{\mu}_2} \wedge \hat{e}^{\overline{\mu}_3} \\ &+ e^{-(2\alpha+\beta)\rho} \left(\frac{1}{2!} H^{(2)\overline{a}}_{\overline{\mu}_1\overline{\mu}_2\overline{i}_1} + H^{(1)\overline{a}}_{\overline{\mu}_1\overline{i}_1\overline{i}_2} \tilde{A}^{\overline{i}_2}_{\overline{\mu}_2}\right) \hat{e}^{\overline{\mu}_1} \wedge \hat{e}^{\overline{\mu}_2} \wedge \hat{e}^{\overline{i}_1} \\ &+ \frac{1}{2!} e^{-(\alpha+2\beta)\rho} H^{(1)\overline{a}}_{\overline{\mu}_1\overline{i}_1\overline{i}_2} \hat{e}^{\overline{\mu}_1} \wedge \hat{e}^{\overline{i}_1} \wedge \hat{e}^{\overline{i}_2} \\ &= e^{-3\alpha\rho} \left(H^{(3)\overline{a}} - H^{(2)\overline{a}}_{\overline{i}_1} \wedge A^{(1)\overline{i}_1} + \frac{1}{2} H^{(1)\overline{a}}_{\overline{i}_1\overline{i}_2} \wedge \tilde{A}^{(1)\overline{i}_1} \wedge \tilde{A}^{(1)\overline{i}_2}\right) \\ &+ e^{-(2\alpha+\beta)\rho} \left(H^{(2)\overline{a}}_{\overline{i}_1} + H^{(1)\overline{a}}_{\overline{i}_1\overline{i}_2} \wedge \tilde{A}^{(1)\overline{i}_2}\right) \wedge \hat{e}^{\overline{i}_1} \\ &+ \frac{1}{2!} e^{-(\alpha+2\beta)\rho} H^{(1)\overline{a}}_{\overline{i}_1\overline{i}_2} \wedge \hat{e}^{\overline{i}_1} \wedge \hat{e}^{\overline{i}_2}.\end{aligned} \tag{5.12}$$

Note that this is equivalent to equation (3.50) of chapter 3 and in the last line we have expressed the field strengths $H^{(k)}$ in coordinate free notation, where

$$H^{(k)} = \frac{1}{k!} H^{(k)}_{\overline{\mu}_1\overline{\mu}_2\ldots\overline{\mu}_k} \hat{e}^{\overline{\mu}_1} \wedge \hat{e}^{\overline{\mu}_1} \wedge \ldots \wedge \hat{e}^{\overline{\mu}_k}. \tag{5.13}$$

Comparing this with an expansion of the ten dimensional field strength $\hat{H}^{(3)\overline{a}}$ in terms of the dimensionally reduced field strengths in tangent frame, given by

$$\hat{H}^{\overline{a}} = \hat{H}^{(3)\overline{a}} + \hat{H}^{(2)\overline{a}}_{\bar{i}_1} \wedge \hat{e}^{\bar{i}_1} + \hat{H}^{(1)\overline{a}}_{\bar{i}_1\bar{i}_2} \wedge \hat{e}^{\bar{i}_1} \wedge \hat{e}^{\bar{i}_2}, \tag{5.14}$$

we find that the components of the dimensionally reduced field strengths in tangent frame are

$$\hat{H}^{(3)\overline{a}} = e^{-3\alpha\rho}\left(H^{(3)\overline{a}} - H^{(2)\overline{a}}_{\bar{i}_1} \wedge A^{(1)\bar{i}_1} + \frac{1}{2}H^{(1)\overline{a}}_{\bar{i}_1\bar{i}_2} \wedge \tilde{A}^{(1)\bar{i}_1} \wedge \tilde{A}^{(1)\bar{i}_2}\right), \tag{5.15}$$

$$\hat{H}^{(2)\overline{a}}_{\bar{i}_1} = e^{-(2\alpha+\beta)\rho}\left(H^{(2)\overline{a}}_{\bar{i}_1} + H^{(1)\overline{a}}_{\bar{i}_1\bar{i}_2} \wedge \tilde{A}^{(1)\bar{i}_2}\right), \tag{5.16}$$

$$\hat{H}^{(1)\overline{a}}_{\bar{i}_1\bar{i}_2} = e^{-(\alpha+2\beta)\rho}\left(H^{(1)\overline{a}}_{\bar{i}_1\bar{i}_2}\right). \tag{5.17}$$

Dimensional reduction of the $d = 10$ five-form field strength proceeds similarly, one finds

$$\begin{aligned}\hat{F} = e^{-5\alpha\rho}\Bigg(&F^{(5)} - F^{(4)}{}_{\bar{i}_1} \wedge A^{(1)\bar{i}_1} + \frac{1}{2}F^{(3)}{}_{\bar{i}_1\bar{i}_2} \wedge A^{(1)\bar{i}_1} \wedge A^{(1)\bar{i}_2} \\ &-\frac{1}{3!}F^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3} \wedge A^{(1)\bar{i}_1} \wedge A^{(1)\bar{i}_2} \wedge A^{(1)\bar{i}_3}\Bigg) \\ + e^{-(4\alpha+\beta)\rho}&\left(F^{(4)}{}_{\bar{i}_1} + F^{(3)}{}_{\bar{i}_1\bar{i}_2} \wedge A^{\bar{i}_2} + \frac{1}{2!}F^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3} \wedge A^{(1)\bar{i}_2} \wedge A^{(1)\bar{i}_3}\right) \wedge \hat{e}^{\bar{i}_1} \\ + e^{-(3\alpha+2\beta)\rho}&\left(\frac{1}{2!}F^{(3)}{}_{\bar{i}_1\bar{i}_2} - F^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3} \wedge A^{(1)\bar{i}_3}\right) \wedge \hat{e}^{\bar{i}_1} \wedge \hat{e}^{\bar{i}_2} \\ + e^{-(2\alpha+3\beta)\rho}&\left(\frac{1}{3!}F^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3}\right) \wedge \hat{e}^{\bar{i}_1} \wedge \hat{e}^{\bar{i}_2} \wedge \hat{e}^{\bar{i}_3},\end{aligned} \tag{5.18}$$

where

$$F^{(5)} = dC^{(4)} + \frac{1}{2}\epsilon_{ab}B^{(2)\overline{a}} \wedge dB^{(2)\overline{b}}, \tag{5.19}$$

$$F^{(4)}{}_{\bar{i}_1} = dC^{(3)}{}_{i_1} + \frac{1}{2}\epsilon_{ab}\left(B^{(2)\overline{a}} \wedge dB^{(1)\overline{b}}_{\bar{i}_1} - B^{(1)\overline{a}}_{\bar{i}_1} \wedge dB^{(2)\overline{b}}\right), \tag{5.20}$$

$$F^{(3)}{}_{\bar{i}_1\bar{i}_2} = dC^{(2)}{}_{\bar{i}_1\bar{i}_2} + \frac{1}{2}\epsilon_{\overline{a}\overline{b}}\left(B^{(2)\overline{a}} \wedge dB^{(0)\overline{a}} + B^{(0)\overline{a}}_{\bar{i}_1\bar{i}_2} \wedge dB^{(2)\overline{b}} + 2B^{(1)\overline{a}}_{[\bar{i}_1} \wedge dB^{(1)\overline{b}}_{\bar{i}_2]}\right), \tag{5.21}$$

$$F^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3} = dC^{(1)}_{\bar{i}_1\bar{i}_2\bar{i}_3} - \frac{3}{2}\epsilon_{\overline{a}\overline{b}}\left(B^{(1)\overline{a}}_{[\bar{i}_1} \wedge dB^{(0)\overline{b}}_{\bar{i}_2\bar{i}_3]} - B^{(0)\overline{b}}_{[\bar{i}_1\bar{i}_2} \wedge dB^{(1)\overline{a}}_{\bar{i}_3]}\right). \tag{5.22}$$

Again, comparing this with an expansion of the ten dimensional field strength $\hat{F}$ in terms of the dimensionally reduced field strengths in tangent frame leads to the components of the dimensionally reduced five-form field strength in tangent frame being given by

$$\begin{aligned}
\hat{F}^{(5)} &= e^{-5\alpha\rho}\Bigg(F^{(5)} - F^{(4)}{}_{\bar{i}_1} \wedge A^{(1)\bar{i}_1} + \frac{1}{2}F^{(3)}{}_{\bar{i}_1\bar{i}_2} \wedge A^{(1)\bar{i}_1} \wedge A^{(1)\bar{i}_2} \\
&\qquad - \frac{1}{3!}F^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3} \wedge A^{(1)\bar{i}_1} \wedge A^{(1)\bar{i}_2} \wedge A^{(1)\bar{i}_3}\Bigg), \\
\hat{F}^{(4)}{}_{\bar{i}_1} &= e^{-(4\alpha+\beta)\rho}\left(F^{(4)}{}_{\bar{i}_1} + F^{(3)}{}_{\bar{i}_1\bar{i}_2} \wedge A^{\bar{i}_2} + \frac{1}{2!}F^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3} \wedge A^{(1)\bar{i}_2} \wedge A^{(1)\bar{i}_3}\right), \\
\hat{F}^{(3)}{}_{\bar{i}_1\bar{i}_2} &= e^{-(3\alpha+2\beta)\rho}\left(\frac{1}{2!}F^{(3)}{}_{\bar{i}_1\bar{i}_2} - F^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3} \wedge A^{(1)\bar{i}_3}\right), \\
\hat{F}^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3} &= e^{-(2\alpha+3\beta)\rho}\left(\frac{1}{3!}F^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3}\right).
\end{aligned} \tag{5.23}$$

Having dimensionally reduced the five-form field strength $\hat{F}$ we may now implement the self duality condition

$$\hat{F} = *\hat{F}, \tag{5.24}$$

where the Hodge dual in this expression is with respect to both the $d = 7$ background manifold and the $d = 3$ compact internal manifold. The Hodge dual of the five-form field strength $\hat{F}$ in tangent frame is

$$\begin{aligned}
*\hat{F} = &\frac{1}{5!}\hat{F}^{(5)}_{\overline{\mu}_1\overline{\mu}_2\overline{\mu}_3\overline{\mu}_4\overline{\mu}_5}\frac{1}{2!3!}\epsilon^{\overline{\mu}_1\overline{\mu}_2\overline{\mu}_3\overline{\mu}_4\overline{\mu}_5}{}_{\overline{\mu}_6\overline{\mu}_7}\epsilon_{\bar{i}_1\bar{i}_2\bar{i}_3}\hat{e}^{\overline{\mu}_6} \wedge \hat{e}^{\overline{\mu}_7} \wedge \hat{e}^{\bar{i}_1} \wedge \hat{e}^{\bar{i}_2} \wedge \hat{e}^{\bar{i}_3} \\
&+ \frac{1}{4!}\hat{F}^{(4)}{}_{\overline{\mu}_1\overline{\mu}_2\overline{\mu}_3\overline{\mu}_4\bar{i}_1}\frac{1}{3!2!}\epsilon^{\overline{\mu}_1\overline{\mu}_2\overline{\mu}_3\overline{\mu}_4}{}_{\overline{\mu}_5\overline{\mu}_6\overline{\mu}_7}\epsilon^{\bar{i}_1}{}_{\bar{i}_2\bar{i}_3}\hat{e}^{\overline{\mu}_5} \wedge \hat{e}^{\overline{\mu}_6} \wedge \hat{e}^{\overline{\mu}_7} \wedge \hat{e}^{\bar{i}_2} \wedge \hat{e}^{\bar{i}_3} \\
&+ \frac{1}{3!}\hat{F}^{(3)}{}_{\overline{\mu}_1\overline{\mu}_2\overline{\mu}_3\bar{i}_1\bar{i}_2}\frac{1}{4!1!}\epsilon^{\overline{\mu}_1\overline{\mu}_2\overline{\mu}_3}{}_{\overline{\mu}_4\overline{\mu}_5\overline{\mu}_6\overline{\mu}_7}\epsilon^{\bar{i}_1\bar{i}_2}{}_{\bar{i}_3}\hat{e}^{\overline{\mu}_4} \wedge \hat{e}^{\overline{\mu}_5} \wedge \hat{e}^{\overline{\mu}_6} \wedge \hat{e}^{\overline{\mu}_7} \wedge \hat{e}^{\bar{i}_3} \\
&+ \frac{1}{2!}\hat{F}^{(2)}_{\overline{\mu}_1\overline{\mu}_2\bar{i}_1\bar{i}_2\bar{i}_3}\frac{1}{5!}\epsilon^{\overline{\mu}_1\overline{\mu}_2}{}_{\overline{\mu}_3\overline{\mu}_4\overline{\mu}_5\overline{\mu}_6\overline{\mu}_7}\epsilon^{\bar{i}_1\bar{i}_2\bar{i}_3}\hat{e}^{\overline{\mu}_3} \wedge \hat{e}^{\overline{\mu}_4} \wedge \hat{e}^{\overline{\mu}_5} \wedge \hat{e}^{\overline{\mu}_6} \wedge \hat{e}^{\overline{\mu}_7}.
\end{aligned} \tag{5.25}$$

Suppressing background manifold coordinate indices through adopting coordinate free notation the Hodge dual of the five-form field strength is

$$\begin{aligned}
\hat{F} = &\hat{F}^{(5)}\frac{1}{3!}\epsilon_{\bar{i}_1\bar{i}_2\bar{i}_3} \wedge \hat{e}^{\bar{i}_1} \wedge \hat{e}^{\bar{i}_2} \wedge \hat{e}^{\bar{i}_3} \\
&+ *\hat{F}^{(4)}{}_{\bar{i}_1}\frac{1}{2!}\epsilon^{\bar{i}_1}{}_{\bar{i}_2\bar{i}_3} \wedge \hat{e}^{\bar{i}_2} \wedge \hat{e}^{\bar{i}_3} \\
&+ *\hat{F}^{(3)}{}_{\bar{i}_1\bar{i}_2}\epsilon^{\bar{i}_1\bar{i}_2}{}_{\bar{i}_3} \wedge \hat{e}^{\bar{i}_3} \\
&+ *\hat{F}^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3}\epsilon^{\bar{i}_1\bar{i}_2\bar{i}_3},
\end{aligned} \tag{5.26}$$

where the Hodge dual on the right hand side in the above expression is purely with respect to the $d = 7$ background manifold. Comparing the components of the internal tangent space basis elements of $*\hat{F}$ and $\hat{F}$ as given in equation (5.26) we find

$$\frac{1}{3!}\epsilon_{\bar{i}_1\bar{i}_2\bar{i}_3} * \hat{F}^{(5)} = \hat{F}^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3}, \tag{5.27}$$

$$\frac{1}{2!}\epsilon^{\bar{i}_1}{}_{\bar{i}_2\bar{i}_3} * \hat{F}^{(4)}{}_{\bar{i}_1} = \hat{F}^{(3)}{}_{\bar{i}_2\bar{i}_3}, \tag{5.28}$$

where again the Hodge dual is purely with respect to the $d = 7$ background manifold. Substituting for $\hat{F}^{(5)}$ and $\hat{F}^{(4)}{}_{\bar{i}_1}$ using equations (5.27), (5.28) in the dimensionally reduced action has the net effect of doubling the coefficient of the two and three form field strengths arising from the dimensional reduction of the five-form field strength in $d = 10$, thus providing these terms with the standard canonical factor of $\frac{1}{2}$ usually associated with the two derivative terms in the effective action.

Expanding the ten dimensional fields in the IIB action (5.1) in terms of the $d = 7$ fields as given in (5.6), (5.12), (5.18) and collecting the $d = 7$ three-form field strengths, two-form field strengths, derivatives of the scalars and the $d = 7$ curvature R, one may write the dimensionally reduced $d = 7$ action as

$$S_{d=7} = S_R - S_{Scalar} - S_{Two-form} - S_{Three-form}, \tag{5.29}$$

where

$$\begin{aligned}
S_R &= \frac{1}{2\kappa_{10}} \int R \wedge *1, \\
S_1 &= \frac{1}{2\kappa_{10}} \int \Bigg(S_{\bar{i}_1}{}^{\bar{i}_2} \wedge * S_{\bar{i}_1}{}^{\bar{i}_2} + \frac{1}{2} d\rho \wedge * d\rho + \frac{1}{2} d\phi \wedge * d\phi + \frac{1}{2} e^{2\phi} d\chi \wedge * d\chi \\
&\qquad + \frac{1}{2}\frac{1}{2!} e^{-\frac{20}{3}\alpha\rho} \epsilon_{\bar{a}\bar{b}} H^{(1)\bar{a}}_{\bar{i}_1\bar{i}_2} \wedge * H^{(1)\bar{b}\;\bar{i}_3\bar{i}_4} \Bigg), \\
S_2 &= \frac{1}{2\kappa_{10}} \int \Bigg(\frac{1}{2}\frac{1}{3!} e^{8\alpha\rho} F^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3} \wedge * F^{(2)\bar{i}_1\bar{i}_2\bar{i}_3} \\
&\qquad + \frac{1}{2} e^{\frac{8}{3}\alpha\rho} \left(H^{(2)\bar{a}}_{\bar{i}_1} + H^{(1)\bar{a}}_{\bar{i}_1\bar{i}_2} \wedge A^{(1)\bar{i}_2} \right) \wedge * \left(H^{(2)\bar{b}\;\bar{i}_1} + H^{(2)\bar{b}\;\bar{i}_1}{}_{\bar{i}_2} \wedge A^{(1)\bar{i}_2} \right) \\
&\qquad + \frac{1}{2} \tilde{F}_{(2)\bar{i}} \wedge \tilde{F}^{(2)\bar{j}} \Bigg), \\
S_3 &= \frac{1}{2\kappa_{10}} \int \Bigg(\frac{1}{2} e^{-4\alpha\rho} \epsilon_{\bar{a}\bar{b}} \left(H^{(3)\bar{a}} - H^{(2)\bar{a}}_{\bar{i}_1} \wedge A^{(1)\bar{i}_1} + \frac{1}{2} H^{(1)\bar{a}}_{\bar{i}_1\bar{i}_2} \wedge A^{(1)\bar{i}_1} \wedge A^{(1)\bar{i}_2} \right) \\
&\qquad \wedge * \left(H^{(3)\bar{b}} - H^{(2)\bar{b}}_{\bar{i}_1} \wedge A^{(1)\bar{i}_1} + \frac{1}{2} H^{(1)\bar{b}}_{\bar{i}_1\bar{i}_2} \wedge A^{(1)\bar{i}_1} \wedge A^{(1)\bar{i}_2} \right) \\
&\qquad + \frac{1}{2}\frac{1}{2!} e^{\frac{8}{3}\alpha\rho} \left(F^{(3)}{}_{\bar{i}_1\bar{i}_2} - 2F^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3} \wedge A^{(1)\bar{i}_3} \right) \wedge * \left(F^{(3)\bar{i}_1\bar{i}_2} - 2F^{(2)\bar{i}_1\bar{i}_2}{}_{\bar{i}_3} \wedge A^{(1)\bar{i}_3} \right) \Bigg).
\end{aligned} \tag{5.30}$$

5.2 $SL(5)$ Formulation

We now seek to write the dimensionally reduced type IIB effective action in manifestly $SL(5, \mathbb{Z})$ invariant form. To do so, we must carefully select a group element $E \in SL(5, \mathbb{R})/SO(5, \mathbb{R})$ parameterised by the type IIB dilaton and axion, in addition to the scalar fields arising from the reduction of the three form field strengths and those parameterising the internal metric G. The derivatives of the scalars then appear in the action through the symmetric part of the Cartan form constructed from the group element E, while the dimensionally reduced two and three form field strengths transforming in the **5** and $\overline{\mathbf{10}}$ of $SL(5,\mathbb{Z})$ respectively may be converted to non-linear representations of $SL(5,\mathbb{Z})$ via the group element E. The $d=7$ metric is an $SL(5,\mathbb{Z})$ invariant, therefore the $d=7$ curvature R transforms trivially under $SL(5,\mathbb{Z})$.

5.2.1 Scalar Sector

It is expedient for us to choose an upper triangular coset representative $E \in SL(5)/SO(5)$, defined by

$$E = \begin{pmatrix} e^{-2\alpha\rho-\frac{\phi}{2}} & e^{-2\alpha\rho+\frac{\phi}{2}}\chi & e^{\frac{4}{3}\alpha\rho}C^l e_l{}^{\bar{j}} \\ 0 & e^{-2\alpha\rho+\frac{\phi}{2}} & e^{\frac{4}{3}\alpha\rho}B^l e_l{}^{\bar{j}} \\ & & \\ 0 & 0 & e^{\frac{4}{3}\alpha\rho}e_i{}^{\bar{j}} \\ & & \end{pmatrix}. \tag{5.31}$$

The inverse coset element $E^{-1} \in SL(5)/SO(5)$ is then

$$E^{-1} = \begin{pmatrix} e^{2\alpha\rho+\frac{\phi}{2}} & e^{2\alpha\rho+\frac{\phi}{2}}\chi & C'^j \\ 0 & e^{2\alpha\rho-\frac{\phi}{2}} & B'^j \\ & & \\ 0 & 0 & e^{-\frac{4}{3}\alpha\rho}e_{\bar{i}}{}^j \\ & & \end{pmatrix}, \tag{5.32}$$

where

$$\begin{aligned} C'^j =& e^{2\alpha\rho+\frac{\phi}{2}}\left(-C^j+\chi B^j\right), \\ B'^j =& -e^{2\alpha\rho-\frac{\phi}{2}}B^j. \end{aligned} \tag{5.33}$$

The components of the Cartan form $E^{-1}dE$ are

$$\begin{aligned} \left(E^{-1}dE\right)_{\bar{1}}{}^{\bar{1}} &= -2\alpha d\rho - \frac{1}{2}d\phi, \\ \left(E^{-1}dE\right)_{\bar{1}}{}^{\bar{2}} &= e^{\phi}d\chi, \\ \left(E^{-1}dE\right)_{\bar{2}}{}^{\bar{2}} &= -2\alpha\rho + \frac{1}{2}d\phi, \\ \left(E^{-1}dE\right)_{\bar{1}}{}^{\bar{k}} &= e^{\frac{10}{3}\alpha\rho+\frac{\phi}{2}}\left(dC^{\bar{k}} - \chi dB^{\bar{k}}\right), \\ \left(E^{-1}dE\right)_{\bar{2}}{}^{\bar{k}} &= e^{\frac{10}{3}\alpha\rho-\frac{\phi}{2}}\left(dB^{\bar{k}}\right), \\ \left(E^{-1}dE\right)_{\bar{i}}{}^{\bar{k}} &= \frac{4}{3}\alpha\left(d\rho\right)\delta_{\bar{i}}{}^{\bar{k}} + e_{\bar{i}}{}^{j}de_j{}^{\bar{k}}. \end{aligned} \tag{5.34}$$

The components of the symmetric part of the Cartan form EdE^{-1} under the Cartan involution, which we shall denote by $\mathcal{S}$, are then given by

$$\begin{aligned}
\mathcal{S}_{\overline{1}}{}^{\overline{1}} &= -2\alpha d\rho - \frac{1}{2}d\phi, \\
\mathcal{S}_{\overline{1}}{}^{\overline{2}} &= \frac{1}{2}e^{\phi}d\chi, \\
\mathcal{S}_{\overline{2}}{}^{\overline{2}} &= -2\alpha\rho + \frac{1}{2}d\phi, \\
\mathcal{S}_{\overline{1}}{}^{\overline{k}} &= \frac{1}{2}e^{\frac{10}{3}\alpha\rho+\frac{\phi}{2}}\left(dC^{\overline{k}} - \chi dB^{\overline{k}}\right), \\
\mathcal{S}_{\overline{2}}{}^{\overline{k}} &= \frac{1}{2}e^{\frac{10}{3}\alpha\rho-\frac{\phi}{2}}\left(dB^{\overline{k}}\right), \\
\mathcal{S}_{\overline{i}}{}^{\overline{k}} &= \frac{4}{3}\alpha\left(d\rho\right)\delta_{\overline{i}}{}^{\overline{k}} + e_{\overline{i}}{}^{j}de_{j}{}^{\overline{k}}.
\end{aligned} \tag{5.35}$$

Tracing over the symmetric part of the Cartan form, we find

$$\begin{aligned}
\mathcal{S}_{\overline{I}}{}^{\overline{J}} \wedge *\mathcal{S}_{\overline{J}}{}^{\overline{I}} &= \frac{40}{3}\alpha^2 d\rho\wedge *d\rho + \frac{1}{2}d\phi\wedge *d\phi + \frac{1}{2}e^{2\phi}d\chi\wedge *d\chi \\
&+ \frac{1}{2}e^{\frac{20}{3}\alpha\rho+\phi}g_{ml}\left(dC^l\wedge *dC^m - 2\chi dB^l\wedge *dC^m + |\tau|^2 dB^l\wedge *dB^m\right) \\
&= \frac{1}{2}d\rho\wedge *d\rho + \frac{1}{2}d\phi\wedge *d\phi + \frac{1}{2}e^{2\phi}d\chi\wedge *d\chi + S_{\overline{i}}{}^{j}\wedge *S_{\overline{j}}{}^{i} \\
&+ \frac{1}{2}\frac{1}{2!}e^{\frac{20}{3}\alpha\rho+\phi}\delta_{\overline{a}\overline{b}}H^{(1)\overline{a}}_{\overline{i}_1\overline{i}_2}\wedge *H^{(1)\overline{b}\ \overline{i}_1\overline{i}_2}
\end{aligned} \tag{5.36}$$

where, in the last line, we have used $\alpha^2 = \frac{3}{80}$ and rewritten the derivatives of the NS-NS and R-R dimensionally reduced fields, B_{ij} and C_{ij} respectively, as a non-linear realisation of $SL(2,\mathbb{R})$. So $\mathcal{S}_{\overline{I}}{}^{\overline{J}} \wedge *\mathcal{S}_{\overline{J}}{}^{\overline{I}}$ contains the scalar part of the action S_1 in (5.30).

5.2.2 Two-form Field Strengths

The 2-form fields originate from the 3-form field strengths, the 5-form field strength and the ten dimensional scalar curvature. We will define a one-form gauge field $\tilde{A}^{(1)}_{IJ}$ that lies in the **10** of $SL(5,\mathbb{Z})$, by

$$\begin{aligned}
\tilde{A}^{(1)}_{12} &= \frac{1}{3!}\epsilon^{lmn}\tilde{C}^{(1)}_{lmn}, \\
\tilde{A}^{(1)}_{1j} &= -\tilde{C}_j, \\
\tilde{A}^{(1)}_{2j} &= -\tilde{B}_j, \\
\tilde{A}^{(1)}_{ij} &= -A_{ij},
\end{aligned} \tag{5.37}$$

where $A_{ij} = \epsilon_{kij}A^k$. The two-form field strength constructed by taking $K_{IJ} = d\tilde{A}^{(1)}_{IJ}$ also transforms in the **10** of $SL(5,\mathbb{Z})$. One then finds that the two-form field strength $\mathcal{F}$ defined by

$$\mathcal{F}_{\overline{I}\overline{J}} = E^{-1}{}_{\overline{I}}{}^{K}E^{-1}{}_{\overline{J}}{}^{L}K_{KL}, \tag{5.38}$$

is a non-linear representation of $SL(5,\mathbb{R})$ and transforms in a representation of $SO(5,\mathbb{R})$, in this instance the adjoint representation. The components of the non-linearly realised two-form field strength $\mathcal{F}$ are

$$
\begin{aligned}
\mathcal{F}_{\overline{12}} &= e^{4\alpha\rho}\left(\frac{1}{3!}\epsilon^{lmn}d\tilde{C}^{(1)}_{lmn} + \epsilon_{ab}B^{(0)aj}d\tilde{B}^{(1)b}_{j} + \frac{1}{2}\epsilon_{ab}B^{(0)ai}B^{(0)bj}\tilde{F}_{ij}\right),\\
\mathcal{F}_{\overline{a}\overline{j}} &= -e^{-\frac{2}{3}\alpha\rho}\epsilon_{\overline{a}\overline{b}}\left(\tilde{H}^{(1)\overline{b}}_{\overline{j}} + B^{(0)b\ k}\tilde{F}_{\overline{j}k}\right),\\
\mathcal{F}_{\overline{i}\overline{j}} &= e^{-\frac{8}{3}\alpha\rho}\tilde{F}_{\overline{i}\overline{j}}.
\end{aligned}
\tag{5.39}
$$

The two-form field strength part of the action S_2 in (5.30) may then be written

$$
\frac{1}{2\kappa_7^2}\int\frac{1}{4}\mathcal{F}_{\overline{IJ}}\wedge*\mathcal{F}^{\overline{IJ}} = \frac{1}{2}\mathcal{F}_{\overline{12}}\wedge*\mathcal{F}^{\overline{12}} + \frac{1}{2}cF_{\overline{a}\overline{j}}\wedge*\mathcal{F}^{\overline{a}\overline{j}} + \frac{1}{4}\mathcal{F}_{\overline{i}\overline{j}}\wedge*\mathcal{F}^{\overline{i}\overline{j}}. \tag{5.40}
$$

To show this we will first make the following field redefinitions in the two-form field strength part of the action S_2

$$
B^{(1)\overline{a}}_{\overline{i}_1} = \tilde{B}^{(1)\overline{a}}_{\overline{i}_1} - B^{(1)\overline{a}}_{\overline{i}_1\overline{i}_2}\wedge A^{(1)\overline{i}_2}, \tag{5.41}
$$

and

$$
C^{(1)}_{\overline{i}_1\overline{i}_2\overline{i}_3} = \tilde{C}^{(1)}_{\overline{i}_1\overline{i}_2\overline{i}_3} - \frac{3}{2}\epsilon_{\overline{a}\overline{b}}B^{(1)\overline{a}}_{[\overline{i}_1}\wedge B^{(0)\overline{b}}_{\overline{i}_2\overline{i}_3]} + \frac{3}{2}\epsilon_{\overline{a}\overline{b}}B^{(1)\overline{a}}_{[\overline{i}_1\overline{i}_2}\wedge B^{(1)\overline{b}}_{\overline{i}_3]\overline{i}_4}\wedge\tilde{A}^{(1)\overline{i}_4}. \tag{5.42}
$$

The two-form field strengths arising from dimensional reduction of the $d=10$ three form field strengths then become

$$
\begin{aligned}
H^{(2)\overline{a}}_{\overline{i}_1} + H^{(1)\overline{a}}_{\overline{i}_1\overline{i}_2}\wedge A^{(1)\overline{i}_2} &= d\tilde{B}^{(1)\overline{a}}_{\overline{i}_1} - dB^{(0)\overline{a}}_{\overline{i}_1\overline{i}_2}\wedge\tilde{A}^{(1)\overline{i}_1} - B^{(0)\overline{a}}_{\overline{i}_1\overline{i}_2}\wedge d\tilde{A}^{(1)\overline{i}_2} + dB^{(0)\overline{a}}_{\overline{i}_1\overline{i}_2}\wedge\tilde{A}^{(1)\overline{i}_2}\\
&= \tilde{H}^{(2)\overline{a}}_{\overline{i}_1} - B^{(0)\overline{a}}_{\overline{i}_1\overline{i}_2}\wedge\tilde{F}^{(2)\overline{i}_2},
\end{aligned}
\tag{5.43}
$$

where we have defined $\tilde{H}^{(2)\overline{a}} = d\tilde{B}^{(1)\overline{a}}$. One may also show, through the use of identities $A.9$ and $A.13$, that the two-form field strengths originating from the $d=10$ five-form field strength become

$$
F^{(2)}_{\overline{i}_1\overline{i}_2\overline{i}_3} = d\tilde{C}^{(1)}_{\overline{i}_1\overline{i}_2\overline{i}_3} + 3\epsilon_{\overline{a}\overline{b}}B^{(0)\overline{a}}_{[\overline{i}_1\overline{i}_2}\wedge d\tilde{B}^{(1)\overline{b}}_{\overline{i}_3]} - \frac{3}{2}\epsilon_{\overline{a}\overline{b}}B^{(0)\overline{a}}_{[\overline{i}_1\overline{i}_2}\wedge B^{(0)\overline{b}}_{\overline{i}_3]\overline{i}_4}\wedge\tilde{F}^{(2)\overline{i}_4}. \tag{5.44}
$$

The two-form part of the action S_2 is then

$$
\begin{aligned}
S_2 = \frac{1}{2\kappa_7^2}&\left(\int\frac{1}{2}\frac{1}{3!}e^{8\alpha\rho}F^{(2)}_{\overline{i}_1\overline{i}_2\overline{i}_3}\wedge*F^{(2)\overline{i}_1\overline{i}_2\overline{i}_3}\right.\\
&+\frac{1}{2}e^{\frac{8}{3}\alpha\rho}\delta_{\overline{a}\overline{b}}\left(\tilde{H}^{(2)\overline{a}}_{\overline{i}_1} - B^{(0)\overline{a}}_{\overline{i}_1\overline{i}_2}\wedge\tilde{F}^{(2)\overline{i}_2}\right)\wedge*\left(\tilde{H}^{(2)\overline{b}\ \overline{i}_1} + B^{(0)\overline{b}\ \overline{i}_1}{}_{\overline{i}_2}\wedge\tilde{F}^{(2)\overline{i}_2}\right)\\
&\left.+\frac{1}{2}\tilde{F}_{(2)\overline{i}_1}\wedge*\tilde{F}^{(2)\overline{i}_1}\right).
\end{aligned}
\tag{5.45}
$$

Examining the terms in $\mathcal{L}_{Two-form}$ and $\frac{1}{4}\mathcal{F}_{\overline{IJ}} \wedge *\mathcal{F}^{\overline{IJ}}$ while making extensive use of the identities $(A.9)$, $(A.13)$, one finds

$$\begin{aligned}
\frac{1}{2}\mathcal{F}_{\overline{12}} \wedge *\mathcal{F}^{\overline{12}} &= \frac{1}{2}\frac{1}{3!}e^{8\alpha\rho}F^{(2)}_{\bar{i}_1\bar{i}_2\bar{i}_3} \wedge *F^{(2)\bar{i}_1\bar{i}_2\bar{i}_3}, \\
\frac{1}{2}\mathcal{F}_{\overline{aj}} \wedge *\mathcal{F}^{\overline{aj}} &= \frac{1}{2}e^{\frac{8}{3}\alpha\rho}\delta_{\overline{ab}}\left(\tilde{H}^{(2)\overline{a}}_{\bar{i}_1} - B^{(0)\overline{a}}_{\bar{i}_1\bar{i}_2} \wedge \tilde{F}^{(2)\bar{i}_2}\right) \wedge *\left(\tilde{H}^{(2)\overline{b}\ \bar{i}_1} + B^{(0)\overline{b}\ \bar{i}_1}{}_{\bar{i}_2} \wedge \tilde{F}^{(2)\bar{i}_2}\right), \\
\frac{1}{4}\mathcal{F}_{\overline{ij}} \wedge *\mathcal{F}^{\overline{ij}} &= \frac{1}{2}\tilde{F}_{(2)\bar{i}_1} \wedge *\tilde{F}^{(2)\bar{i}_1}.
\end{aligned} \tag{5.46}$$

So the two-form part of the action S_2 may indeed be written as $\frac{1}{4}\mathcal{F}_{\overline{IJ}} \wedge *\mathcal{F}^{\overline{IJ}}$.

5.2.3 Three-Form Field Strengths

The $d = 7$ three form field strengths arise from the dimensional reduction of the $d = 10$ three form field strengths and five-form field strengths. Defining the two-form gauge field $B^{(2)I}$ transforming in the $\overline{\mathbf{5}}$ of $SL(5, \mathbb{Z})$, by

$$\begin{aligned}
B^{(2)1} &= \tilde{B}^{(2)}, \\
B^{(2)2} &= \tilde{C}^{(2)}, \\
B^{(2)i} &= \frac{1}{2!}\epsilon^{ijk}C^{(2)}_{4jk},
\end{aligned} \tag{5.47}$$

where $\tilde{B}^{(2)}$, $\tilde{C}^{(2)}$ are the redefined NS-NS and R-R two-form gauge fields, respectively, and $C^{(2)}_{4jk}$ are the components of the dimensionally reduced four-form gauge field C_4. The three form field strength given by $G^{(3)I} = dB^{(2)I}$ also transforms in the $\overline{\mathbf{5}}$ of $SL(5, \mathbb{Z})$. Converting the three form field strength $G^{(3)I}$ transforming in the $\overline{\mathbf{5}}$ of $SL(5, \mathbb{Z})$ to a non-linear representation $\mathcal{H}^{\overline{I}}$ transforming under $SO(5, \mathbb{R})$ by taking

$$\mathcal{H}^{\overline{I}} = E_J{}^{\overline{I}}G^{(3)J}, \tag{5.48}$$

it is then believed that the three form field strength part of the dimensionally reduced action S_3 may be written

$$S_3 = \frac{1}{2\kappa_7}\int \frac{1}{2}\delta_{\overline{IJ}}\left(\mathcal{H}^{\overline{I}} - \frac{1}{8}\epsilon^{\overline{IJKLM}}\mathcal{F}_{\overline{JK}}\mathcal{A}_{\overline{LM}}\right) \wedge *\left(\mathcal{H}^{\overline{N}} - \frac{1}{8}\epsilon^{\overline{NPQRS}}\mathcal{F}_{\overline{PQ}}\mathcal{A}_{\overline{RS}}\right), \tag{5.49}$$

where $\epsilon^{\overline{IJKLM}}$ are the components of the $SO(5, \mathbb{R})$ invariant completely antisymmetric tensor and $\mathcal{A}_{\overline{IJ}}$ are the components of the non-linear representation constructed from the two-form gauge field A defined in (5.37). This expression reproduces the three form part of the dimensionally reduced three form field strength denoted $\hat{H}^{(3)\overline{a}}$ in equation (5.17), after a field redefinition in the dimensionally reduced action of the type given in equation (5.42). However, we have yet to confirm that the three forms arising from the dimensionally reduced five-form field strengths are contained in S_3. To do so one needs to find a field redefinition similar to that of equation (5.42) for the dimensionally reduced four-form gauge field $C^{(2)}_{\bar{i}_1\bar{i}_2}$ such that the dimensionally reduced three form part of the five-form field strength matches that given by S_3.

5.3 Constraints on Higher Derivative Terms

Let us assume that the two derivative terms in the dimensionally reduced action of type IIB string theory are reproduced by the $d = 7$ curvature R, the symmetric part of the $SL(5,\mathbb{R})/SO(5,\mathbb{R})$ Cartan forms $\mathcal{S}$ and the non-linear representations of the two and three form field strengths as constructed in the previous section. At next to leading order, the ten dimensional type IIB effective action contains a term of the form

$$\int d^{10}x\sqrt{-\hat{g}}\Phi_{SL(2)}\hat{R}^4, \tag{5.50}$$

where $\Phi_{SL(2)}$ is the $SL(2,\mathbb{Z})$ Eisenstein series with $s = \frac{3}{2}$ and $\hat{R}^4$ is a specific contraction of ten dimensional Riemann tensors. Dimensionally reducing this term on a three torus gives

$$\int d^7x\sqrt{-g}\Phi_{SL(2)}e^{-6\alpha\rho}\left(R^4 + f\left(F, \partial_\mu\rho, S_\mu\right)\right), \tag{5.51}$$

where R^4 is the $d = 7$ equivalent of the specific contraction of ten dimensional Riemann tensors and f is an eighth order polynomial in the two-form field strength F, the derivatives of the volume modulus of the torus ρ and the symmetric part of the $SL(3,\mathbb{R})/SO(3,\mathbb{R})$ Cartan forms S_μ, with contractions between the spacetime and internal indices dependent on the specific contraction of the ten dimensional Riemann tensors. Now, consider the $d = 7$ R^4 term in (5.51) with coefficient $\Phi_{SL(2)}e^{-6\alpha\rho}$. Since R^4 and $\Phi_{SL(2)}$ are invariant while the volume modulus of the torus ρ is covariant under an $SL(5,\mathbb{Z})$ transformation this term seemingly breaks the $SL(5,\mathbb{Z})$ symmetry of the effective action of type IIB string theory compactified on a three torus. However, we know that this is not the full story, instead we expect to have to construct higher order terms in the dimensionally reduced effective action from the $d = 7$ curvature R, the symmetric part of the $SL(5,\mathbb{R})/SO(5,\mathbb{R})$ Cartan forms $\mathcal{S}$ and the non-linear representations of the two and three form field strengths with a coefficient function that transforms as an $SL(5,\mathbb{Z})$ automorphic form $\Phi_{SL(5)}$. Therefore, the resolution to the apparent problem that the dimensionally reduced R^4 term breaks the $SL(5,\mathbb{Z})$ symmetry is that the $\Phi_{SL(2)}e^{-6\alpha\rho}R^4$ term is contained in the $SL(5,\mathbb{Z})$ invariant term $\Phi_{SL(5)}R^4$. In the next chapter we will see that this puts strong constraints on the $\Phi_{SL(5)}$ automorphic form and in fact may be generalised to any dimension $d \geq 3$ and any higher order term in the effective action.

Now let us consider the remaining terms in (5.51) of the form $\Phi_{SL(2)}e^{-6\alpha\rho}f\left(F, \partial_\mu\rho, S_\mu\right)$. In the large volume limit of the three torus, which is equivalent to $\rho \to -\infty$, terms of the form $\Phi_{SL(2)}e^{-6\alpha\rho}f\left(F, \partial_\mu\rho, S_\mu\right)$ along with $\Phi_{SL(2)}e^{-6\alpha\rho}R^4$ must combine to produce the ten dimensional higher derivative term $\Phi_{SL(2)}\hat{R}^4$. The $SL(5,\mathbb{Z})$ completion of terms of the form $\Phi_{SL(2)}e^{-6\alpha\rho}f\left(F, \partial_\mu\rho, S_\mu\right)$

is given by $SL(5,\mathbb{Z})$ invariant terms constructed out of non-linear representations of the two-form field strength $\mathcal{F}$ and the symmetric part of the $SL(5,\mathbb{R})/SO(5,\mathbb{R})$ Cartan forms $\mathcal{S}$ with a coefficient function that transforms as an $SL(5,\mathbb{Z})$ automorphic form $\Phi'_{SL(5)}$. However, in the large volume limit of the three torus, terms of the form $\Phi'_{SL(5)} f\left(F, \partial_\mu \rho, S_\mu\right)$ that combine with $\Phi_{SL(5)} R^4$ to produce the ten dimensional $\Phi_{SL(2)} R^4$ term also give rise to ten dimensional terms of the form $\Phi_{SL(2)} \hat{f}\left(\hat{H}^{\overline{a}}, \hat{F}, P\right)$, where $\Phi_{SL(2)}$ is the automorphic form appearing as the coefficient function in the original ten dimensional $\hat{R}^4$ higher order term and $\hat{f}$ is a function of the ten dimensional three form field strengths $\hat{H}^{\overline{a}}$, five-form field strength $\hat{F}$ and the symmetric part of the $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ Cartan forms P. So, one might speculate that given a higher order term in the effective action of type IIB string theory in ten dimensions constructed out of some polynomial in the curvature R, the three form field strengths $\hat{H}^{\overline{a}}$, five-form field strength $\hat{F}$ and the symmetric part of the $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ Cartan forms P with coefficient function that transforms as an $SL(2,\mathbb{Z})$ automorphic form $\Phi_{SL(2)}$ we will find a set of ten dimensional higher order terms constructed out of various combinations of the ten dimensional fields with the same $SL(2,\mathbb{Z})$ automorphic form.

The argument that given the higher derivative $\Phi_{SL(2)} \hat{R}^4$ term in the type IIB effective action in ten dimensions one should expect to find a set of related but different eight derivative terms constructed from the ten dimensional curvature, $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ Cartan forms, non-linearly realised three form field strengths and five-form field strengths with the automorphic form $\Phi_{SL(2)}$ depends purely on the mixing of the ten dimensional fields in the seven dimensional $SL(5,\mathbb{Z})$ multiplets and makes no reference to the explicit structure of the automorphic form. Therefore, it is natural to extend this argument and conjecture that the existence of a higher derivative term $\Phi_{SL(2)}\mathcal{O}$ in the effective action of type IIB string theory in ten dimensions, where $\mathcal{O}$ is an l derivative polynomial in the ten dimensional curvature, $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ Cartan forms, non-linearly realised three form field strengths and five-form field strengths, implies the existence of a set of higher derivative terms that are a set of related but different l derivative polynomials in the ten dimensional curvature, $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ Cartan forms, non-linearly realised three form field strengths and five-form field strengths, in the effective action possessing the same $SL(2,\mathbb{Z})$ automorphic form $\Phi_{SL(2)}$ as the original term.

6 Constraints on Type IIB and M-theory Automorphic Forms

This chapter is based on reference [28] and develops on an observation made at the end of chapter 5. Namely that the automorphic form $\Phi_{SL(5)}$ of the type IIB R^4 term in $d = 7$ dimensions must contain the term arising from the dimensional reduction of the $d = 10$ type IIB R^4 term and extends it to all dimensions $d \geq 3$ and all higher derivative terms possessing invariant automorphic forms. In particular we will dimensionally reduce a generic higher derivative term of type IIB string theory on an n torus to $d = 10 - n$ dimensions. When one does this one finds that each term in the d dimensional effective action contains a factor of the form $e^{\sqrt{2}\vec{w}\cdot\vec{\phi}}$ for some vector $\vec{w}$. The fields $\vec{\phi}$ are the diagonal components of the metric, which encode the volume and other radii moduli of the torus, as well as any scalar fields in the higher dimensional theory such as the dilaton. Carrying this out for the supergravity theory, that is the lowest energy terms, we find that the vectors $\vec{w}$ that appear are the roots of E_{n+1}. Indeed, this provides the fastest way to see that there is very likely to be an E_{n+1} symmetry of the lower dimensional theory. In references [38, 39] this was carried out for a generic higher derivative term of the effective action and one found not roots but weights of E_{n+1}. This in itself was evidence for an E_{n+1} symmetry in the higher derivative corrections to string theory and the appearance of weights rather than roots was interpreted as evidence for automorphic forms as they involve weights of E_{n+1}, this may be contrasted with the scalar fields parameterising the E_{n+1}/H coset that appear with the roots of E_{n+1}.

We carry out the calculation in a more streamlined manner than in references [38, 39] and generalise to any dimension $d \geq 3$. We assemble the fields of the lower dimensional theory, that occur with spacetime derivatives, into representations of the d-dimensional duality group E_{n+1}. We show that the higher derivative terms can be written as powers of the E_{n+1} covariant field strengths along with additional factors of the dilaton and volume which are the remnants of the above $e^{\sqrt{2}\vec{w}\cdot\vec{\phi}}$ factors. We find that the left-over weight has a simple universal form for any term. For terms that arise at tree level in string perturbation in ten dimensions we find $\vec{w} = s\vec{\Lambda}^{n+1}$, where $s = (l_T - 2)/4$ with l_T counting the number of derivatives and $\vec{\Lambda}^{n+1}$ the fundamental weight dual to $\vec{\alpha}_{n+1}$ (see Figure 9). The observation of [39] is that these additional factors must come from an automorphic form and therefore we are led to conclude that the automorphic form which multiplies a given higher derivative term involves the weight $\vec{\Lambda}^{n+1}$. Moreover, for Eisenstein-like automorphic forms the leading order behaviour is given by $e^{-\sqrt{2}s\vec{\phi}\cdot\vec{\Lambda}^H}$ (for example see [39, 40]), where $\vec{\Lambda}^H$ is the highest weight of the representation used to construct the automorphic form. Thus our results suggest that the higher derivative terms always include an automorphic form built from a representation with highest weight $\vec{\Lambda}^{n+1}$. We also perform a similar analysis in M-theory and see that the weight is (using the same labeling of the E_{n+1} diagram), $\vec{w} = s\vec{\Lambda}^{n-1}$ with $s = (l_T - 2)/6$. However it is important to note that we are in effect considering a particular limit and other representations could also appear but be subdominant in that limit.

6.1 Dimensional Reduction of Type IIB Higher Derivative Terms

In this section we will study the dimensional reduction of the higher derivative corrections of IIB string theory. Our metric compactification ansatz is given by

$$d\hat{s}^2 = e^{2\alpha\rho}ds^2 + e^{2\beta\rho}G_{ij}(dx^i + A^i_\mu dx^\mu)(dx^j + A^j_\mu dx^\mu), \tag{6.1}$$

where G_{ij} has unit determinant and the constants α and β are defined in equations (3.12), (3.13) and ensure that if one starts in D-dimensional Einstein frame, the resulting theory in d dimensions is in Einstein frame with a standard normalisation for the kinetic term of the scalar ρ which controls the volume of the torus. We have labeled the coordinates by $(x^\mu, x^i), \mu = 0, 1, \ldots, d-1;\ i = d, \ldots, D-1$. In the above equation $G_{ij} = e_i{}^{\overline{k}} e_j{}^{\overline{l}} \delta_{\overline{k}\overline{l}}$ and $e_i{}^{\overline{k}}$ is a vielbein but subject to $\det e = 1$. We adopt the convention that $i, j, k, \ldots$ are world indices and $\overline{i}, \overline{j}, \overline{k}, \ldots$ are tangent indices.

As is well-known, dimensionally reducing Einstein gravity on a torus leads to a theory that possesses an $SL(n,\mathbb{R})$ symmetry. In particular, the degrees of freedom of gravity associated with the torus, apart from the graviphotons, enter the lower dimensional theory through a non-linear realization of $SL(n,\mathbb{R})$ with local subgroup $SO(n)$. The latter is the Cartan involution invariant subgroup, *i.e.* $\tau(SL(n)) = SO(n)$.

Using the local subgroup we can bring the $SL(n)$ group element to the form

$$g_{SL(n)}(\xi_{SL(n)}) = e^{\sum_{\underline{\alpha}>0} E_{\underline{\alpha}}\chi_{\underline{\alpha}}} e^{-\frac{1}{\sqrt{2}}\underline{\phi}\cdot\underline{H}}, \tag{6.2}$$

where $\underline{H}$ forms the Cartan subalgebra, $E_{\underline{\alpha}}$ are positive root generators (when $\underline{\alpha} > 0$) of $SL(n,\mathbb{R})$ respectively and $\xi_{SL(n)}$ collectively denotes the fields $\chi_{\underline{\alpha}}$ and $\underline{\phi}$. In fact the terms which contain $g_{SL(n)}(\xi_{SL(n)})$ alone are built out of the Cartan forms

$$g^{-1}_{SL(n)}\partial_\mu g_{SL(n)} = S_{SL(n)\mu} + Q_{SL(n)\mu}\ , \tag{6.3}$$

where $S_{SL(n)\mu}$ and $Q_{SL(n)\mu}$ are symmetric and anti-symmetric in $\overline{i}$ and $\overline{j}$ respectively corresponding to the decomposition of the Cartan forms into those for $SO(n)$, *i.e.* $Q_{SL(n)\mu}$, and its compliment.

In what follows we will construct the dimensionally reduced theory in such a way that its $SL(n,\mathbb{R})$ symmetry is manifest. To begin with we wish to find an expression for the inverse vielbein making use of the discussion of non-linear realisations. Let us denote the the fundamental highest weights of $SL(n)$ by $\underline{\lambda}^i$. The representation with highest weight $\underline{\lambda}^1$ corresponds to the vector representation, with a single lowered index. We denote the states of this representation by $|\psi>= \psi_i|\underline{\mu}^i>$ where $\underline{\mu}^i$ are the weights in the root string of $\underline{\lambda}^1$, which we denote by $[\underline{\lambda}^1]$. From this linear representation we can construct the non-linear representation as described in appendix $B.3$ by taking

$$|\varphi(\xi)>=\sum \varphi_i|\vec{\mu}^i>=L(g^{-1}_{SL(n)}(\xi))|\psi>=e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{H}}e^{\sum_{\vec{\alpha}>0}-\chi_{\vec{\alpha}}E_{\vec{\alpha}}}|\psi>\ , \tag{6.4}$$

where

$$\varphi_i = D(g^{-1}_{SL(n)}(\xi_{SL(n)}))_i{}^j\psi_j\ . \tag{6.5}$$

Under an $SL(n)$ transformation this state transforms under a local $SO(n)$ and we may identify the change from ψ_i to φ_i as the familiar conversion from world to tangent indices using the inverse vielbein. The matrix element of $g^{-1}_{SL(n)}$ in the vector representation is therefore given by

$$(e^{-1})_{\bar{i}}{}^j = D(g^{-1}_{SL(n)}(\xi_{SL(n)}))_i{}^j\ . \tag{6.6}$$

The right-hand end of equation (6.4) contains the factor $e^{\frac{1}{\sqrt{2}}\underline{\phi}\cdot[\underline{\lambda}^1]}$. Thus we find that the inverse vielbein $e_i{}^{\bar{j}}$ contains factors of $e^{-\frac{1}{\sqrt{2}}\underline{\phi}\cdot[\underline{\lambda}^1]}$.

The dimensionally reduced theory will involve corrections that contain field strengths of the form $\mathcal{F}_{\mu_1\ldots\mu_p i_1\ldots i_k}$, where $i_1,\ldots,i_k$ are world volume indices of the torus (there can also be $SL(2)$ indices which we address below). However, we can choose to work with tangent, rather than world, indices in the torus directions by using the inverse vielbein $(e^{-1})_{\bar{i}}{}^j$. Following the same argument we just used above, this can be viewed as the conversion of the linear rank k antisymmetric representation of $SL(n,\mathbb{R})$ into a non-linear representation of $SL(n,\mathbb{R})/SO(n)$ whose indices rotate under $SO(n)$. Thus we start from the linear representation $\sum_{i_1,\ldots i_k} F_{\mu_1\ldots\mu_p i_1\ldots i_k}|i_1\ldots i_k,\underline{\lambda}^k>$ and construct the non-linear realisation

$$\sum_{i_1,\ldots i_k}\mathcal{F}^{sl(n)}_{\mu_1\ldots\mu_p\bar{i}_1\ldots\bar{i}_k}|i_1\ldots i_k,\underline{\lambda}^k>=L(g^{-1}_{SL(n)}(\xi))\sum_{i_1,\ldots i_k}F_{\mu_1\ldots\mu_p i_1\ldots i_k}|i_1\ldots i_k,\underline{\lambda}^k>\ . \tag{6.7}$$

We note that we have denoted the field strength with tangent indices by $\mathcal{F}^{sl(n)}_{\mu_1\ldots\mu_p\bar{i}_1\ldots\bar{i}_k}$. Its dependence on the metric of the torus is obtained by acting with $L(g^{-1}_{SL(n)}(\xi))$ on the states $|i_1\ldots i_k,\underline{\lambda}^k>$. Therefore one finds that the fields $\underline{\phi}$ associated with the Cartan subalgebra of $SL(n,\mathbb{R})$ occur in $\mathcal{F}^{SL(n)}_{\mu_1\ldots\mu_p\bar{i}_1\ldots\bar{i}_k}$ through the factor $e^{\frac{1}{\sqrt{2}}\underline{\phi}\cdot[\underline{\lambda}^k]}$.

6.1.1 $SL(2)$ Formulation of Type IIB Supergravity

As demonstrated in section 3.4.2, the bosonic part of the type IIB effective action, in Einstein frame, can be written

$$
\begin{aligned}
S =& \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left(\hat{R} - \frac{1}{12}\hat{\mathcal{H}}^{\overline{c}}_{\mu\nu\rho}\hat{\mathcal{H}}^{\overline{d}\mu\nu\rho}\delta_{\overline{c}\overline{d}} + \frac{1}{4}tr\left(\hat{\mathcal{S}}_\mu\hat{\mathcal{S}}^\mu\right)\right) \\
&- \frac{1}{8\kappa^2}\left(\int d^{10}x\sqrt{-g}\frac{1}{5!}\hat{F}_{\mu_1\mu_2\mu_3\mu_4\mu_5}\hat{F}^{\mu_1\mu_2\mu_3\mu_4\mu_5} + \int \epsilon_{\overline{a}\overline{b}}C_4\wedge H_3^{\overline{a}}\wedge H_3^{\overline{b}}\right),
\end{aligned}
\tag{6.8}
$$

where $\hat{R}$ is the scalar curvature, $\hat{\mathcal{H}}^{\overline{c}}_{\mu\nu\rho}$ are the components of non-linearly realised three form field strengths, $\hat{\mathcal{S}}_\mu$ are the components of the symmetric part of the $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ Cartan form under the Cartan involution and $\hat{F}^{\mu_1\mu_2\mu_3\mu_4\mu_5}$ are the components of the type IIB five-form field strength. Making the transition to string frame ${e_\mu}^{\overline{\nu}} = e^{-\frac{\phi}{4}}{(e_S)_\mu}^{\overline{\nu}}$, where ${e_\mu}^{\overline{\nu}}$ and ${(e_S)_\mu}^{\overline{\nu}}$ are the components of the vielbein on the Einstein frame and string frame metric, respectively, the type IIB effective action becomes

$$
\begin{aligned}
S_{IIB} = \int d^{10}x\ \det(e_S)(&e^{-2\phi}\hat{R} - \frac{1}{2}e^{-2\phi}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi \\
&- \frac{1}{2\cdot 3!}\hat{\mathcal{H}}^{\overline{1}}_{\mu_1\mu_2\mu_3}\hat{\mathcal{H}}^{\overline{1}\ \mu_1\mu_2\mu_3} - \frac{1}{2.3!}e^{-2\phi}\hat{\mathcal{H}}^{\overline{2}}_{\mu_1\mu_2\mu_3}\hat{\mathcal{H}}^{\overline{2}\ \mu_1\mu_2\mu_3} \\
&- \frac{1}{2\cdot 5!}\hat{F}_{\mu_1\ldots\mu_5}\hat{F}^{\mu_1\ldots\mu_5}).
\end{aligned}
\tag{6.9}
$$

We note from the factors of e^ϕ that occur that $\hat{\mathcal{H}}^{\overline{1}}_{1\mu_1\mu_2\mu_3}$, χ and $\hat{F}_{\mu_1\ldots\mu_5}$ are in the R-R sector and $g_{\mu\nu}$, ϕ, and $\hat{\mathcal{H}}^{\overline{2}}_{1\mu_1\mu_2\mu_3}$ in the NS-NS sector.

Let us now consider higher derivative terms. It will be useful for what follows to use a hat to denote a ten-dimensional quantity and suppress any spacetime indices. The higher derivative corrections in the IIB theory in ten dimensions can be written as a polynomial in the Riemann tensor $\hat{R}$, $SL(2)/SO$ Cartan forms $\hat{\mathcal{S}}$, rank three field strength $\hat{\mathcal{H}}^{\overline{a}}_3$ and rank five field strength $\hat{F}$ with coefficients that are $SL(2)$ automorphic forms. The generic term has the form

$$
\int d^{10}x\ \det(\hat{e})\partial^{\hat{l}_0}\hat{R}^{\frac{\hat{l}_R}{2}}(\hat{\mathcal{S}}_{\mu_1 1})^{\hat{l}_1}(\hat{\mathcal{S}}_{\mu_1 2})^{\hat{l}'_1}(\hat{\mathcal{H}}^{\overline{1}}_{\mu_1\ldots\mu_3})^{\hat{l}_3}(\hat{\mathcal{H}}^{\overline{2}}_{1\mu_1\ldots\mu_3})^{\hat{l}'_3}(\hat{F}_{\mu_1\ldots\mu_5})^{\hat{l}'_5}\hat{\Phi}_{sl_2},
\tag{6.10}
$$

where $\hat{\mathcal{S}}_{\mu_1 1}$ and $\hat{\mathcal{S}}_{\mu_1 2}$ are the NS-NS and R-R sector parts of $\mathcal{S}$ respectively, while $\hat{\Phi}_{SL(2)}$ is a suitable automorphic form. As is well known the higher order corrections involve instantons and other solitonic objects and due to the quantisation conditions on the charges only the $SL(2,\mathbb{Z})$ part of the $SL(2,\mathbb{R})$ symmetry survives. The automorphic form depends on τ that is ϕ and χ. We will be mainly interested in the e^ϕ dependence and we denote the leading dependence of $\hat{\Phi}_{SL(2)}$ on ϕ by $\hat{\Phi}_{SL(2)} \sim e^{-\hat{s}\phi}$.

It will be instructive to compute the e^ϕ dependence of the above ten dimensional higher derivative correction in string frame. The transition from Einstein frame to string frame is given by

$\hat{e} = e^{-\frac{\phi}{4}}\hat{e}_s$. We find that the above term leads to the factor

$$e^{\frac{\phi}{4}(\hat{l}_0+\hat{l}_R+\hat{l}_1+5\hat{l}'_1+\hat{l}_3+5\hat{l}'_3+5\hat{l}'_5-10-4\hat{s})} \ . \tag{6.11}$$

Note that we have used a prime to denote contributions from R-R fields. At order g in perturbation theory we have the contribution $e^{\phi(2g-2)}$ and so we conclude that for a perturbative contribution

$$\hat{s} = \frac{1}{4}(\hat{l}_0 + \hat{l}_R + \hat{l}_1 + 5\hat{l}'_1 + \hat{l}_3 + 5\hat{l}'_3 + 5\hat{l}'_5 - 2 - 8g) \ . \tag{6.12}$$

6.1.2 Reduction of Higher Derivative Type IIB Terms

We are interested in the dimensional reduction of ten dimensional higher derivative corrections of IIB string theory, that is terms as given in equation (6.10), on an n torus to $d = 10-n$ dimensions. As explained above, by working with the non-linear realisations we can formulate the result with a manifestly $SL(2)\otimes SL(n)$ symmetry. The resulting building blocks in d dimensions are the Riemann tensor R which is an $SL(2)\otimes SL(n)$ singlet, the derivatives of the scalars which belong to the Cartan forms of $SL(2)\otimes SL(n)$, $\mathcal{S}_{SL(2)\otimes SL(n)}$ and objects which are non-linear representations of $SL(2)\otimes SL(n)$. As mentioned above the latter arise if one works with "tangent space" quantities. These objects generically denoted by $\mathcal{F}$ are related to the usual field strengths, which transform linearly under $SL(2)\otimes SL(n)$, to the non-linearly realised objects F, by the generic equation

$$|\mathcal{F}_{SL(2)\otimes SL(n)}>= L(g^{-1}_{SL(2)\otimes SL(n)})|F>\equiv e^{\frac{1}{\sqrt{2}}(\underline{\phi}\cdot\underline{H}+\phi H)}e^{-(\sum_{\underline{\alpha}>0}E_{\underline{\alpha}}\chi_{\underline{\alpha}}+E\chi)}|F> \ . \tag{6.13}$$

These $\mathcal{F}$ transform by field dependent $SO(n)\otimes SO(2)$ transformations and so it is easy to construct invariants using the Kronecker delta symbol. The $Q_{SL(2)\otimes SL(n)}$ component of the Cartan forms only enters when we find derivatives of the above objects where it plays the role of a connection.

We are particularly interested of the dependence in the dimensionally reduced action on the scalars ϕ, ρ and $\underline{\phi}$ which we assemble into the $n+1$-vector

$$\vec{\phi} = (\phi, \rho, \underline{\phi})\,. \tag{6.14}$$

The dependence on ϕ and $\underline{\phi}$, which are the Cartan fields associated with $SL(2)\otimes SL(n)$, occurs only inside the objects $\mathcal{F}_{SL(2)\otimes SL(n)}$. The exception is the ϕ dependence that arises from the ten dimensional automorphic form $\hat{\Phi}_{SL(2)}$. The dependence on ρ arises from the dimensional reduction of the vielbeins using the metric ansatz of equation (6.1) as was described in references [38, 39]. The $\det\hat{e}$ factor in the action leads to a factor of $e^{(d\alpha+n\beta)\rho} = e^{2\alpha\rho}$ while $\mathcal{F}_{SL(2)\otimes SL(n)\ \mu_1\ldots\mu_p\bar{i}_1\ldots\bar{i}_k}$ leads to the factor $e^{-\rho(p\alpha+k\beta)}$. To give a concrete example with l factors of the latter field strength we find the generic term

$$\int d^{10}x \det \hat{e}(\hat{\mathcal{F}}^{SL(2)}_{\mu_1\ldots\mu_q})^l \sim \sum_{p+k=q} \int d^dx \det e(\mathcal{F}_{SL(2)\otimes SL(n)\ \mu_1\ldots\mu_p\bar{i}_1\ldots\bar{i}_k})^l e^{(2\alpha - l(p\alpha+k\beta))\rho} . \tag{6.15}$$

The powers of e^ρ associated with any other terms are also easily calculated.

The dimensional reduction of any term in the effective action of equation (6.8) leads to terms that contain the derivatives of the scalars, vielbein and gauge fields multiplied by factors of the form $e^{\sqrt{2}\vec{w}\cdot\vec{\phi}}$ for some $n+1$-vector $\vec{w}$:

$$\vec{w} = (w, \kappa, \underline{w}) . \tag{6.16}$$

The first and third entries w and $\underline{w}$ arise from the behaviour of the fields under the $SL(2)\otimes SL(n)$ and can be read off from the action of $g_{SL(2)\otimes SL(n)}$ on the linearly realised representation using equation (6.13). The second entry simply records the powers of $e^{\sqrt{2}\rho}$ that arise after dimensional reduction as just discussed.

For every factor of $F^{SL(2)}_{\mu_1\mu_2\mu_3}$ that occurs one finds a corresponding factor of $e^{\frac{1}{\sqrt{2}}\phi[\mu]}$, where $[\mu] = \{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\}$ are the weights that appear in the fundamental representation of $SL(2,\mathbb{R})$. In particular the NS-NS and R-R field strengths come with the factors $e^{-\frac{\phi}{2}}$ and $e^{\frac{\phi}{2}}$ respectively.

In what follows it will be advantageous to also consider the dual version of certain fields. Let us consider a two-derivative term in the effective action of the form

$$\int d^dx \det e(\mathcal{F}_{SL(2)\otimes SL(n)\mu_1\ldots\mu_p\bar{i}_1\ldots\bar{i}_k})^2 e^{(2\alpha-2(p\alpha+k\beta)\rho} , \tag{6.17}$$

where $\mathcal{F}_{SL(2)\otimes SL(n)} = g^{-1}_{SL(2)\otimes SL(n)}F$, $F = dA$. We can introduce the dual field strength $\mathcal{F}^D_{SL(2)\otimes SL(n)\mu_1\ldots\mu_q\bar{i}_1\ldots\bar{i}_s}$ defined by $\mathcal{F}^D_{SL(2)\otimes SL(n)} = g^{-1}_{SL(2)\otimes SL(n)}dA^D$ where $p+q=d$ and $k+s=n$. We then impose the Bianchi identity of $F = dA$ by adding to the action the term

$$\begin{aligned}\int d^dx \epsilon^{\mu_1\ldots\mu_p\nu_1\ldots\nu_q}\epsilon^{i_1\ldots\bar{i}_kj_1\ldots\bar{i}_s}F_{SL(2)\otimes SL(n)\mu_1\ldots\mu_p\bar{i}_1\ldots\bar{i}_k}F^D_{SL(2)\otimes SL(n)\nu_1\ldots\nu_q\bar{j}_1\ldots\bar{j}_s}\\ = \int d^dx \epsilon^{\mu_1\ldots\mu_p\nu_1\ldots\nu_q}\epsilon^{i_1\ldots\bar{i}_kj_1\ldots\bar{i}_s}\mathcal{F}_{SL(2)\otimes SL(n)\mu_1\ldots\mu_p\bar{i}_1\ldots\bar{i}_k}\mathcal{F}^D_{SL(2)\otimes SL(n)\nu_1\ldots\nu_q\bar{j}_1\ldots\bar{j}_s} ,\end{aligned} \tag{6.18}$$

where $F^D = dA^D$ and in the second line with have used the fact that $\det(g_{SL(2)\otimes SL(n)}) = 1$. Note that if $\mathcal{F}_{SL(2)\otimes SL(n)}$ and $\mathcal{F}^D_{SL(2)\otimes SL(n)}$ have $SL(2)$ indices then an additional factor of ϵ^{ab} is needed in (6.18).

We can now view $\mathcal{F}_{SL(2)\otimes SL(n)}$ as an unconstrained field and integrate it out. Taking its equation of motion implies that

$$\mathcal{F}_{SL(2)\otimes SL(n)\mu_1\ldots\mu_p\bar{i}_1\ldots\bar{i}_k} \sim \epsilon^{i_1\ldots\bar{i}_kj_1\ldots\bar{i}_s}\epsilon^{\mu_1\ldots\mu_p\nu_1\ldots\nu_q}\mathcal{F}^D_{SL(2)\otimes SL(n)\nu_1\ldots\nu_q\bar{j}_1\ldots\bar{j}_s}e^{2(-\alpha+(p\alpha+k\beta))\rho} \tag{6.19}$$

We will assume that we can use this lowest order dualisation equation in the higher order corrections. Therefore, for each factor of $\mathcal{F}_{SL(2)\otimes SL(n)\mu_1\dots\mu_p\bar{i}_1\dots\bar{i}_k}$ we find in the higher derivative terms

$$\begin{aligned}\mathcal{F}_{SL(2)\otimes SL(n)\mu_1\dots\mu_p\bar{i}_1\dots\bar{i}_k}e^{-(p\alpha+k\beta)\rho} &\sim \mathcal{F}^D_{SL(2)\otimes SL(n)\nu_1\dots\nu_q\bar{j}_1\dots\bar{j}_s}e^{-2\alpha+(p\alpha+k\beta)\rho} \\ &\sim \mathcal{F}^D_{SL(2)\otimes SL(n)\nu_1\dots\nu_q\bar{j}_1\dots\bar{j}_s}e^{-(q\alpha+s\beta)\rho} \ .\end{aligned} \tag{6.20}$$

In the last step we have used equations (3.12) and (3.13) for the constants α and β. Hence, we get the same result if we use the original field or we use the dual field provided we take into account the correct number of indices. The reader may check this in specific cases including that of the graviphoton which first appears when reducing the Riemann tensor with a field strength that carries a single upper i index.

6.1.3 The E_{n+1} symmetry in d Dimensions

As discussed in the last section the dimensional reduction of the IIB theory including its higher derivative corrections, on an n torus leads to a formulation in which the $SL(2)\otimes SL(n)$ symmetry is manifest. However, the IIB supergravity theory when dimensionally reduced to $d = 10 - n$ dimensions actually possess an E_{n+1} symmetry, of which a discrete subgroup is preserved in the quantum theory. Evidence for this conjecture has been obtain in a variety of works such as [15, 23, 24, 27, 31–48]. The Dynkin diagram of E_{n+1} suited to the IIB theory is given by

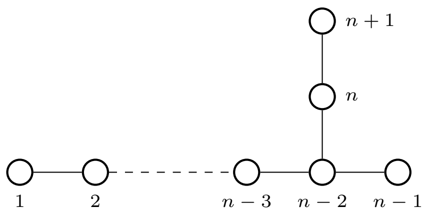

Figure 9: Dynkin diagram for E_{n+1} with type IIB labeling

The relevant $SL(2)\otimes SL(n)$ subalgebra of E_{n+1} is found by deleting the node labeled n in the Dynkin diagram of Figure 9. The $SL(2)$ factor is just the $SL(2)$ symmetry of the IIB supergravity theory and arises from the node labeled $n+1$, while the $SL(n)$ symmetry is part of the gravity symmetry of the ten dimensional theory that now belongs to the torus and corresponds to the nodes labeled 1 to $n-1$.

These features are particularly apparent when one considers the E_{11} formulation of the IIB theory [66, 67]. The E_{n+1} Dynkin diagram emerges from the E_{11} Dynkin diagram, given just below, by deleting the node d to find the algebra $SL(d)\otimes E_{n+1}$. The nodes labeled 1 to 9 of the E_{11} Dynkin diagram in figure 10 are called the gravity line as they are associated with ten dimensional gravity. After the deletion of the node d, this line gives rise to $SL(d)\otimes SL(n)$ which

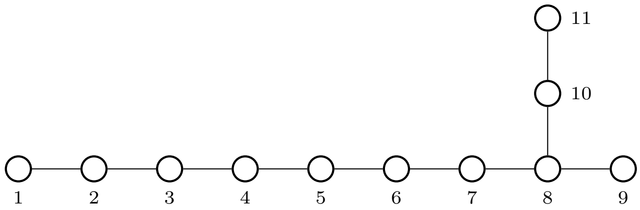

Figure 10: E_{11} Dynkin diagram with type IIB labeling

is associated with gravity in d dimensions and the $SL(n)$ of the now internal E_{n+1} symmetry.

As already mentioned if one computes the weights $\vec{w}$ that arise from the dimensional reduction of the IIB supergravity theory using the techniques given in the last section one readily finds that they are the weights of E_{n+1}. While this is a strong indication of an underlying E_{n+1} symmetry the detailed dimensional reduction is required to prove the existence of this symmetry in d dimensions. In this process one finds that the $SL(2) \otimes SL(n)$ representations that the fields belong to collect up to form a representation of E_{n+1}. It will be essential to understand how the representations of E_{n+1} that occur decompose into representations of $SL(2) \otimes SL(n)$ as this will allow us to compare the E_{n+1} formulation of the higher derivative corrections with that arising from dimensional reduction from ten dimensions. It is from this comparison that we will be able to deduce some properties of the automorphic form in d dimensions. The review [82] on U-duality discusses E_{n+1} representations but here we will need the explicit form for the weights.

d	E_{n+1}	$\tau\,(E_{n+1})$	F_2	F_3	F_4	F_5
10	$SL(2)$	$SO(2)$		$\mathbf{2}$		$\mathbf{1}$
8	$SL(3)\times SL(2)$	$SO(3)\times SO(2)$	$(\overline{\mathbf{3}},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	
7	$SL(5)$	$SO(5)$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	
6	$SO(5,5)$	$SO(5)\times SO(5)$	$\mathbf{16}$	$\mathbf{10}$		
5	E_6	$USp(8)$	$\mathbf{27}$			
4	E_7	$SU(8)$	$\mathbf{56}$			
3	E_8	$SO(16)$				

Table 2: E_{n+1}, $\tau(E_{n+1})$ and representations of the field strengths

The scalars, denoted ξ_E, in d dimensions belong to a non-linear realisation of E_{n+1} with local subgroup $\tau(E_{n+1})$ where $\tau(G)$ denotes the Cartan involution invariant subgroup of G. These local subgroups are given in Table 2. Given a group element $g_E(\xi_E)$ of E_{n+1} we can use the local transformation $\tau(E_{n+1})$ to cast it in the form

$$g_E(\xi_E) = e^{\sum_{\vec{\alpha}>0} E_{\vec{\alpha}}\chi_{\vec{\alpha}}} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{H}} \tag{6.21}$$

where $E_{\vec{\alpha}}$ are the positive root and $\vec{H}$ the Cartan subalgebra generators of E_{n+1}. The fields $\vec{\phi}$ and $\chi_{\vec{\alpha}}$ are the scalar fields of the theory which we have denoted collectively by ξ_E. The dynamics of the scalars are constructed, as usual, out of the Cartan form $g_E^{-1}dg_E = \mathcal{S}_E + Q_E$, where Q_E lies in the Lie-algebra of $\tau(E_{n+1})$.

The gauge fields transforms as linear representations of E_{n+1}; their representations are given in Table 2. Note that care must be taken for $d/2$-form field strengths as these generally only fill out E_{n+1} representations if their electromagnetic duals are also included. However, it is desirable to use the scalar fields ξ_E to convert the fields strengths F_E which belong to linear realisations of E_{n+1}, into tensors denoted $\mathcal{F}_{\mathcal{E}}$ which transform non-linearly under E_{n+1}, using equation $(B.73)$. We may write the relation in the generic form

$$|\mathcal{F}_E >= L(g_E^{-1}(\xi_E))|F_E > \ . \tag{6.22}$$

Under an E_{n+1} transformation these change as

$$|\mathcal{F}_E >\to L(h^{-1})|\mathcal{F}_E >, \tag{6.23}$$

where $h \in I(E_{n+1})$. We can write $g_E(\xi_E) = g_{SL(2)\otimes SL(n)}(\xi_{SL(2)\otimes SL(n)})g'$ where g' contains the Cartan and positive root generators of E_{n+1} which are outside $SL(2)\otimes SL(n)$. Therefore we can write

$$\begin{aligned} |\mathcal{F}_E > &= L(g_E^{-1}(\xi_E))|F > \\ &= L((g')^{-1}L(g^{-1}_{SL(2)\otimes SL(n)})(\xi_{SL(2)\otimes SL(n)})|F > \\ &= \sum_{(\mu,\underline{\lambda})} L((g')^{-1})|\mathcal{F}^{(\mu,\underline{\lambda})}_{SL(2)\otimes SL(n)} > \ . \end{aligned} \tag{6.24}$$

Hence the E_{n+1} non-linear realisations $\mathcal{F}_E$ that appear in the E_{n+1} formulation of the theory can be written as $L((g')^{-1})$ acting on the non-linear realisations $\mathcal{F}^{(\mu,\underline{\lambda})}_{SL(2)\otimes SL(n)}$. The superscript $(\mu,\underline{\lambda})$ are the highest weights of the different $SL(2)\otimes SL(n)$ representations that arise in the decomposition of the linear representation F, that is $F = \sum_{(\mu,\underline{\lambda})} F^{(\mu,\underline{\lambda})}_{SL(2)\otimes SL(n)}$.

We will primarily be interested in the scalar fields associated with the Cartan subalgebra of E_{n+1}. The subalgebra $SL(n)\otimes SL(2)$ has n such fields $\underline{\phi}$ and ϕ which are associated with the nodes $1,\ldots,n-1$ and node $n+1$ of the E_{n+1} Dynkin diagram respectively. The remaining Cartan field in E_{n+1} is ρ and this is associated with the deleted node, that is the node n. Restricting g to the Cartan sub-algebra, denoted $g_c = e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{H}}$ we find that

$$\begin{aligned} L(g_c^{-1})|\vec{\lambda} > &= e^{+\frac{1}{\sqrt{2}}\vec{\Lambda}\cdot\vec{\phi}}|\vec{\Lambda} > \\ &= e^{\frac{1}{\sqrt{2}}(\vec{\Lambda})_n\rho}e^{+\frac{1}{\sqrt{2}}\underline{\lambda}\cdot\underline{\phi}}e^{+\frac{1}{\sqrt{2}}(\vec{\Lambda})_{n+1}\phi}|\vec{\Lambda} > \\ &= e^{\frac{1}{\sqrt{2}}(\vec{\Lambda})_n\rho}L(g^{-1}_{SL(2)\otimes SL(n)c})|\vec{\Lambda} > , \end{aligned} \tag{6.25}$$

when acting on a state in a representation of E_{n+1} with weight $\vec{\Lambda}$. Here $(\vec{\Lambda})_n$ is the nth component of $\vec{\Lambda}$. We are interested in comparing the E_{n+1} formulation of the higher derivative corrections in d dimensions with those obtained by dimensional reduction from ten dimensions, both of which can be written in terms of non-linear realisation of $SL(2) \otimes SL(n)$ symmetry, *i.e.* in terms of $\mathcal{F}^{(\mu,\underline{\lambda})}_{SL(2)\otimes SL(n)}$. Consequently, it is the difference which is of most interest, namely the $e^{-\frac{1}{\sqrt{2}}(\vec{\Lambda})_n\rho}$ factors. In the E_{n+1} formulation these arise by decomposing the E_{n+1} building blocks $\mathcal{F}$ as in equation (6.24) and then using equation (6.25) while in the dimensional reduction they arise from the metric ansatz of equation (6.1).

We assume that the higher derivative corrections to the IIB theory are invariant under a discrete E_{n+1} symmetry. The fields transform in the same way as for the IIB supergravity theory in d dimensions, but under the discrete group. The terms in the d dimensional effective action will be of the generic form

$$\int d^dx \det e \partial^{l_0} R^{\frac{l_R}{2}} \mathcal{S}^{l_1}_{E\ \mu} (\mathcal{F}_{E\mu_1})^{l_1} (\mathcal{F}_{E\mu_1\mu_2})^{l_2} \Phi_E \dots \ , \tag{6.26}$$

where $\mathcal{F}_{E\mu_1}, \dots$ are the E_{n+1} non-linear realisations constructed in equation (6.24) and Φ_E is function of the scalars ξ which transforms under the discrete symmetry as

$$\Phi_E \rightarrow D(h^{-1})\Phi_E \ , \tag{6.27}$$

for $h \in I(E_{n+1})$ and $D(h)$ being in the representation that Φ_E belongs to. However, Φ_E has a non-holomorphic dependence on the scalars and we will refer to it as a non-holomorphic automorphic form.

A formulation of automorphic forms which transform as in equation (6.27) was given in reference [29]. To construct such a non-holomorphic automorphic form for a discrete group G one chooses a linear representation of G denoted $|\psi>$ and considers $|\varphi>= L(g^{-1})|\psi>$ where $g(\xi)$ is an element of G that is subject to the transformations of equation $(B.61)$, that is it is a non-linear realisation and $|\varphi>$ is the non-linear realisation constructed from $|\psi>$ using equation $(B.73)$. The automorphic form is a suitable function of φ. The simplest case is that of a scalar automorphic form that is given by

$$\Phi = \sum_{|\psi>\neq 0} \frac{1}{<\varphi|\varphi>^s} \ . \tag{6.28}$$

For our case $G = E_{n+1}$ and ξ are the scalar fields of the theory, include those associated with Cartan subalgebra which we have labeled by $\vec{\phi} = (\phi, \rho, \underline{\phi})$. To leading order the automorphic form will have a dependence on these scalars which we denote by

$$\Phi_E \sim e^{-\sqrt{2}\vec{\Lambda}_\Phi \cdot \vec{\phi}} \ , \tag{6.29}$$

where $\vec{\Lambda}_\Phi$ is a weight of the representation. For the automorphic form of equation (6.29) $\Phi \sim e^{-\sqrt{2}s\vec{\Lambda}_H\cdot\vec{\phi}}$ where $\vec{\Lambda}_H$ is the highest weight of the representation used to build the automorphic form.

We will want to compare the terms in the effective action of equation (6.26) in their E_{n+1} formulation with those obtained from the dimensional reduction of the higher derivative terms in ten dimensions given in equation (6.10). This will allow us to place restrictions on the automorphic form Φ_E in d dimensions and in particular the weights $\vec{\Lambda}_\Phi$ that can appear in it. For almost all terms this will require the decomposition of the E_{n+1} representations that occur into $SL(2)\otimes SL(n)$ representations.

The simplest examples are terms in the effective action of equation (6.26) that only involve powers of the the Riemann tensor in d dimensions since the Riemann tensor is a singlet of E_{n+1}. This contribution comes from the dimensional reduction of the similar term in ten dimensions, namely that of equation (6.10) with only $\hat{l}_R = l_R$ non vanishing. Since the Riemann tensor, in tangent frame, possess two powers of the inverse vielbein we find a factor of $e^{-2\alpha\rho}$ for each Riemann tensor and a factor of $e^{2\alpha\rho}$ from $\det \hat{e}$. From the automorphic form in ten dimensions we find, at leading order, a factor of $e^{-\hat{s}\phi}$. Thus from dimensional reduction we find in d dimensions the term

$$\int d^d x \det e R^{\frac{l_R}{2}} \Phi_E e^{-\hat{s}\phi-(l_R-2)\alpha\rho} \ . \tag{6.30}$$

Comparing this with the E_{n+1} formulation in d dimensions which is of the form $\int d^d x \det e R^{\frac{l_R}{2}} \hat{\Phi}_E$ we see that the additional factor of ϕ and ρ must arise from the automorphic form Φ_E and so we find that,

$$\vec{\Lambda}_\Phi = \left(\frac{\hat{s}}{\sqrt{2}}, \alpha\frac{(l_R-2)}{\sqrt{2}}, \underline{0}\right) \ . \tag{6.31}$$

From equation (6.12) we have $\hat{s} = \frac{1}{4}(l_R - 2 - 8g)$ and taking the leading contribution at $g = 0$ we conclude that $\vec{\Lambda}_\Phi = \frac{1}{4}(l_R - 2)\vec{\Lambda}^{n+1}$ where $\Lambda^{n+1} = (\frac{1}{\sqrt{2}}, \frac{1}{2x}, \underline{0})$ and we have used the relation $x^{-1} = 4\sqrt{2}\alpha$. Thus the automorphic form has the leading order behaviour $\Phi \sim e^{-\sqrt{2}\frac{1}{4}(l_R-2)\vec{\Lambda}^{n+1}\cdot\vec{\phi}}$. Hence for terms which contain only the Riemann curvature it is straight forward to to compute the leading behaviour of the automorphic form. In what follows we will carry out this calculation for all possible terms, but as we will see this involves some much more sophisticated group theory.

In order to study the remaining terms. We consider the possible building blocks that arise in the dimensional reduction from ten dimensions and compare these with those in the E_{n+1} formulation. As we have explained above the latter can be expressed in terms of non-linear realisations of $SL(2)\otimes SL(n)$ which agree with the same objects found from dimensional reduction. The difference arises from the ρ dependence. To find this difference we must decompose the representations of E_{n+1} into those of $SL(2) \otimes SL(n)$. We do this following the techniques [83–85] developed for the study of the E_{11} symmetry. As mentioned above, deleting the node n in the Dynkin diagram of E_{n+1} results in the algebras $SL(2) \otimes SL(n)$. We may write the simple roots of E_{n+1} as

$$\vec{\alpha}_{n+1} = (\beta_1, 0, 0),\ \vec{\alpha}_n = (0, x, \underline{0}) - \vec{\nu},\ \vec{\alpha}_i = (0, 0, \underline{\alpha}_i),\ i = 1, \dots, n-1, \tag{6.32}$$

where $\vec{\nu} = (\mu_1, 0, \underline{0}) + (0, 0, \underline{\lambda}^{n-2})$. Also, the $\underline{\alpha}_i$ and $\underline{\lambda}^i$ are the simple roots and fundamental weights of $SL(n)$ and $\beta_1 = \sqrt{2}$ and $\mu = \frac{1}{\sqrt{2}}$ the simple root and fundamental weight of $SL(2)$. Demanding that $\vec{\alpha}_n^2 = 2$ we find that $x = \sqrt{\frac{8-n}{2n}} = (4\sqrt{2}\alpha)^{-1}$. The fundamental weights of E_{n+1}, denoted $\vec{\Lambda}^a, a = 1, \ldots, n+1$, satisfy $\vec{\alpha}_a \cdot \vec{\Lambda}^b = \delta_{a,b}$ and are given by

$$\vec{\Lambda}^i = (0, \frac{1}{x}\underline{\lambda}^{n-2} \cdot \underline{\lambda}^i, \underline{\lambda}^i), \qquad \vec{\Lambda}^n = (0, \frac{1}{x}, \underline{0}), \qquad \vec{\lambda}^{n+1} = (\mu, \frac{1}{2x}, \underline{0}). \tag{6.33}$$

Any root of E_{n+1} can be written as

$$\vec{\alpha} = n_c \vec{\alpha}_n + m\vec{\beta}_1 + \sum_i n_i \vec{\alpha}_i = n_c(0, x, 0) - \vec{\Lambda}, \tag{6.34}$$

where $\vec{\Lambda} = n_c\vec{\nu} - \sum_i n_i(0, 0, \underline{\alpha}_i) - m(\beta_1, 0, \underline{0})$. The latter is a weight of $SL(2) \otimes SL(n)$. If a representation of $SL(2) \otimes SL(n)$ occurs in the decomposition of the adjoint representation of E_{n+1} its highest weight must occur as one of the λ's for some positive integers m, n_i and n_c. We refer to the integer n_c as the level and we can analyse the occurrence of highest weights level by level using the techniques of references [83–85]. Clearly, at level zero $i.e$ $n_c = 0$ we have just the adjoint representation of $SL(2) \otimes SL(n)$. The result is that the adjoint representation of E_{n+1} contains the adjoint representation of $SL(2) \otimes SL(n)$ at $n_c = 0$ together with the following highest weight representations of $SL(2) \otimes SL(n)$

$$\begin{array}{cccc} n_c = 1 & n_c = 2 & n_c = 3 & n_c = 4 \\ (\mu, \underline{\lambda}^2) & (0, \underline{\lambda}^4) & (\mu, \underline{\lambda}^6) & (0, \underline{\lambda}^1 + \underline{\lambda}^{n-7}). \end{array} \tag{6.35}$$

Thus the weights in the adjoint representation of E_{n+1} then have the form

$$\begin{array}{c} ([\beta_1], 0, \underline{0}) \ , (0, 0, [\underline{\alpha}_1 + \ldots + \underline{\alpha}_{n_1}]) \ , \ ([\mu_1], x, [\underline{\lambda}^1]) \ , \\ (0, 2x, [\underline{\lambda}^4]) \ , \ ([\mu_1], 3x, [\underline{\lambda}^6]) \ , (0, 4x, [\underline{\lambda}^1 + \underline{\lambda}^{n-7}]) \ . \end{array} \tag{6.36}$$

These correspond to the adjoint of $SL(2) \otimes SL(n)$ at $n_c = 0$ as well as the generators

$$\begin{array}{cccc} n_c = 1 & n_c = 2 & n_c = 3 & n_c = 4 \\ R^{\alpha ij} & R^{i_1 \ldots i_4} & R^{\alpha, i_1 \ldots i_6} & R^{j_1 \ldots j_7 i} \ . \end{array} \tag{6.37}$$

The maximum value of n_c that contributes is $n_c = 1, 2, 2, 3, 4$ for $n = 3, 4, 5, 6, 7$ respectively as is clear from the index structures of the generators. The reader may verify that once the additional negative root generators are included this collection of generators has the correct count of generators for E_{n+1} for $n = 3, \ldots, 7$.

The Cartan forms of E_{n+1} belong to the adjoint representation and so using equation (6.36) we find that the coset component $\mathcal{S}_{E\mu}$ decomposes into the Cartan forms $S_{SL(n) \otimes SL(2)\mu}$ of $SL(2) \otimes$

$SL(n)$ at $n_c = 0$ and

$$\begin{array}{cccc} n_c = 1 & n_c = 2 & n_c = 3 & n_c = 4 \\ S_{SL(2)\otimes SL(n)\mu\alpha ij} & S_{SL(2)\otimes SL(n)\mu i_1\ldots i_4} & S_{\mu\alpha, i_1\ldots i_6} & S_{SL(2)\otimes SL(n)\mu j_1\ldots j_7 i}. \end{array} \tag{6.38}$$

The Cartan form contains the factor $e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\alpha}}$, contained in the g^{-1} part of $g^{-1}\partial_\mu g$, and so using equation (2.43) we find that the level n_c contribution comes with the factor

$$e^{\frac{1}{\sqrt{2}}n_c x\rho} = e^{2\alpha\rho n_c \frac{8-n}{n}} . \tag{6.39}$$

The ten dimensional origin of the first two terms of equation (6.38) is obvious given their index structure and they are contained in the blocks $\mathcal{F}_{SL(2)\otimes SL(n)\alpha\mu ij}$ and $F_{SL(2)\otimes SL(n)\mu i_1\ldots i_4}$ respectively that come from the dimensional reduction of the three form and five-form field strengths respectively. The fourth term of equation (2.44) only occurs for $d = 3$ and $d = 4$ and in these dimensions it arises as the dual of the three from field strength, more precisely the dual of $\mathcal{F}_{SL(2)\otimes SL(n)\ \alpha\nu_1\nu_2\nu_3}$ and $\mathcal{F}_{SL(2)\otimes SL(n)\ \alpha\nu_1\nu_2 i}$ respectively. Alternatively, one can think of the fourth term as arising from the dimensional reduction of the field strength $\mathcal{F}_{SL(2)\alpha\nu_1\ldots\nu_7}$. The final term in equation (6.38) only occurs in $d = 3$ dimensions, that is for E_8, and it arises as the dual of the graviphoton $\partial_{[a}h_{b]}{}^i$. At the end of section (6.1) we showed that calculating the powers e^ρ from the original field, or its dual, gave the same result. As such we will calculate it from the Cartan forms of equation (6.37). We observe that these carry one d dimensional spacetime index and $2n_c$ internal indices and according to the discussion around equation (6.15) we find a factor of

$$e^{-\rho(\alpha+2n_c\beta)} = e^{2\alpha\rho n_c \frac{8-n}{n}} e^{-\alpha\rho} , \tag{6.40}$$

for each contribution.

Thus for each factor of the Cartan form $\mathcal{S}_{E\mu}$ in the d dimensional effective action we find an additional factor of $e^{-\alpha\rho}$ in the dimensionally reduced action compared to the E_{n+1} formulation. This result, taken together with the previous result for factors of the Riemann tensor, is consistent with the rule that for each spacetime derivative in d dimensions we get an additional factor of $e^{-\alpha\rho}$.

To treat the other building blocks in the same way we must learn how to decompose more general representations of E_{n+1} into those of $SL(2) \otimes SL(n)$. To do this we use the technique of reference [74, 75]. If one wants to consider the fundamental representation $\vec{\Lambda}^i$ of E_{n+1} associated with the node labeled i we add a new node, denoted $\star$, to the E_{n+1} Dynkin diagram which is connected to the node labeled i by a single line to construct the Dynkin diagram for an enlarged algebra of rank $n+2$. Deleting the $\star$-node we recover the E_{n+1} Dynkin diagram and the $\vec{\Lambda}^i$ of E_{n+1} is found in the adjoint representation of the enlarged algebra provided we keep only contributions at level $n_\star = 1$. Thus we find the decomposition of the fundamental representation of E_{n+1} into representations of $SL(2)\otimes SL(n)$ by decomposing the adjoint representation of the enlarged algebra but deleting the additional node and keeping only contributions with $n_\star = 1$ and deleting node n but keeping all levels of n_c. The level one states are a representation as the commutator preserves the level and so the commutator of the level zero generators, that is the adjoint representation of E_{n+1}, with the level one states give again level one states. It is the desired representation since the lowest state contains $\underline{\lambda}^i$. For details see references [74, 75].

The weights of the $\vec{\Lambda}^i$ representation of E_{n+1} can be written in the form

$$\left([\mu], n_c x - \frac{1}{x}\underline{\lambda}^{n-2}\cdot\underline{\lambda}^i, [\underline{\lambda}]\right) , \tag{6.41}$$

except for $i = n$ for which it is of the form $([\mu], n_c x - \frac{1}{x}, [\underline{\lambda}])$ Here $(\mu, \underline{\lambda})$ is the highest weight of the $SL(2)\otimes SL(n)$ representation that occurs. We note that $\vec{\nu}\cdot\vec{\Lambda}^i = \frac{2i}{n}$ for $i \leq n-2$, $\vec{\nu}\cdot\vec{\Lambda}^{n-1} = \frac{(n-2)}{n}$ and $\vec{\nu}\cdot\vec{\Lambda}^{n+1} = \frac{1}{2}$.

Next we will treat the two-form field strengths in the d dimensional effective action in a similar way. The one-form gauge field, from which they are constructed, belong to the $\vec{\Lambda}^1$ representation of E_{n+1}. The $\vec{\Lambda}^1$ representation of E_{n+1} decomposes into $SL(2)\otimes SL(n)$ representations as follows

$$\begin{array}{ccccccc} n_c=0 & n_c=1 & n_c=2 & n_c=3 & n_c=4 & n_c=4 & n_c=4 \\ (0,\underline{\lambda}^1) & (\mu,\underline{\lambda}^{n-1}) & (0,\underline{\lambda}^{n-3}) & (\mu,\underline{\lambda}^{n-5}) & (2\mu,\underline{\lambda}^{n-7}) & (\mu,\underline{\lambda}^{n-7}) & (0,\underline{\lambda}^{n-1}+\underline{\lambda}^{n-6}) \\ n_c=5 & & n_c=6 & & n_c=7 & & n_c=8 \\ (\mu,\underline{\lambda}^{n-2}+\underline{\lambda}^{n-7}) & & (0,\underline{\lambda}^{n-4}+\underline{\lambda}^{n-7}) & & (\mu,\underline{\lambda}^{n-6}+\underline{\lambda}^{n-7}) & & (0,\underline{\lambda}^{n-1}+2\underline{\lambda}^{n-7}). \end{array} \tag{6.42}$$

The reader may verify that one finds the correct dimensions of the $\vec{\Lambda}^1$ representation, that is 16, 27, 56 and 248 for $n = 4, 5, 6$ and 7. The weights of the $\vec{\Lambda}^1$ representation are given by

$$(0, \frac{2}{nx}, [\underline{\lambda}^1]) , ([\mu_1], \frac{2}{nx} - x, [\underline{\lambda}^{n-1}]) , (0, \frac{2}{nx} - 2x, [\underline{\lambda}^{n-3}]) , ([\mu_1], \frac{2}{nx} - 3x, [\underline{\lambda}^{n-5}]) , \tag{6.43}$$

$$(2[\mu_1], \frac{1}{2x} - 4x, [\underline{\lambda}^{n-7}]) , (0, \frac{2}{nx} - 4x, [\underline{\lambda}^{n-7}]) , (0, \frac{2}{nx} - 4x, [\underline{\lambda}^1] + [\underline{\lambda}^{n-6}]) , \tag{6.44}$$

$$([\mu_1], \frac{2}{nx} - 5x, [\underline{\lambda}^{n-2} + \underline{\lambda}^{n-7}]) , (0, \frac{2}{nx} - 6x, [\underline{\lambda}^{n-4} + \underline{\lambda}^{n-7}]) , \tag{6.45}$$

$$([\mu_1], \frac{2}{nx} - 7x, [\underline{\lambda}^{n-6} + \underline{\lambda}^{n-7}]) , ([\mu_1], \frac{2}{nx} - 8x, [\underline{\lambda}^1 + 2\underline{\lambda}^{n-7}]) \tag{6.46}$$

These correspond to two-form field strengths take the form

$$\begin{array}{ccccccc} n_c=0 & n_c=1 & n_c=2 & n_c=3 & n_c=4 & n_c=4 & n_c=4 \\ \mathcal{F}^i_{\mu_1 a_2} & \mathcal{F}_{\alpha\mu_1 a_2 i} & \mathcal{F}_{\mu_1 a_2 i_1 i_2 i_3} & \mathcal{F}_{\alpha\mu_1 a_2 i_1\ldots i_5} & \mathcal{F}_{\mu_1 a_2 j, i_1\ldots i_6} & \mathcal{F}_{\mu_1 a_2(\alpha\beta), i_1\ldots i_7} & \mathcal{F}_{\mu_1 a_2 i_1\ldots i_7} , \end{array} \tag{6.47}$$

as well as higher level contributions. Since a two form field strength is dual to a one-form field strength in three dimensions we only study two-form field strengths in dimensions four and above. This corresponds to $n \leq 6$ and so of the above field strengths we only need those at levels $n_c = 3$ and the first term in the above equation at level $n_c = 4$.

We recognise the two-form field strengths of equation (6.47) as the dimensional reduction of the metric, *i.e.* the graviphoton, the three form, the five-form for the first three entries. The fourth entry arises from the dual of the three form in $d = 4$ and $d = 5$ while the only required level four field strength is the dual of the graviphoton.

Decomposing the rank two field strength in their E_{n+1} representation, using equations (6.22) and (6.25), we find the factor

$$e^{\frac{1}{\sqrt{2}}(\frac{\vec{\mu}\cdot\vec{\lambda}^1}{x} - n_c x)\rho} = e^{\frac{2\sqrt{2}\alpha}{n}(-4+n_c(8-n))\rho} , \tag{6.48}$$

for each rank two field strength at level n_c. We observe that the above field strengths have two d-dimensional spacetime indices and $2n_c - 1$ internal indices and so the factor of e^{ρ} that appears when carrying out the dimensional reduction from ten dimensions is

$$e^{-\rho(2\alpha+(2n_c-1)\beta)} = e^{\frac{2\sqrt{2}\alpha}{n}(-4+n_c(8-n))\rho} e^{-\alpha\rho} . \tag{6.49}$$

Evaluating this and comparing with the factor in equation (6.48) we find an additional factor of $e^{-\alpha\rho}$ for each rank two field strength.

We now carry out the same analysis for the rank three field strengths. We need only consider these field strengths in dimensions $d \geq 6$, since in a lower dimension a rank three field strength is dual to a lower rank field strength. This is equivalent to $n \leq 4$. The rank three field strength belong to the $\vec{\Lambda}^{n+1}$ representation of E_{n+1}. One finds that the weights in the $\vec{\Lambda}^{n+1}$ representation of E_{n+1} have the form

$$([\mu_1], \frac{1}{2x}, \underline{0}) , (0, \frac{1}{2x} - x, [\underline{\lambda}^{n-2}]) , ([\mu_1], \frac{1}{2x} - 2x, [\underline{\lambda}^{n-7}]) , \tag{6.50}$$

$$(0, \frac{1}{2x} - 3x, [\underline{\lambda}^{n-1}] + [\underline{\lambda}^{n-5}]) , (0, \frac{1}{2x} - 3x, [\underline{\lambda}^{n-6}]) , ([\beta_1], \frac{1}{2x} - 3x, [\underline{\lambda}^{n-6}]) , \tag{6.51}$$

$$([\mu_1], \frac{1}{2x} - 4x, [\underline{\lambda}^{n-1} + \underline{\lambda}^{n-7}]) , ([\mu_1], \frac{1}{2x} - 4x, [\underline{\lambda}^{n-6} + \underline{\lambda}^{n-2}]) , \tag{6.52}$$

$$(0, \frac{1}{2x} - 5x, [\underline{\lambda}^{n-4} + \underline{\lambda}^{n-6}]) , (0, \frac{1}{2x} - 5x, [\underline{\lambda}^{n-1} + \underline{\lambda}^{n-2} + \underline{\lambda}^{n-7}]) , \tag{6.53}$$

$$([\beta_1], \frac{1}{2x} - 5x, [\underline{\lambda}^{n-3} + \underline{\lambda}^{n-7}]) , \tag{6.54}$$

$$([\mu_1], \frac{1}{2x} - 6x, [\underline{\lambda}^{n-5} + \underline{\lambda}^{n-7}]) , ([\mu_1], \frac{1}{2x} - 6x, [\underline{2}\lambda^{n-6}]) , \tag{6.55}$$

$$([\mu_1], \frac{1}{2x} - 6x, [\underline{\lambda}^{n-1} + \underline{\lambda}^{n-4} + \underline{\lambda}^{n-4}]) \ , \dots . \tag{6.56}$$

The reader may like to verify that one has the correct count of states for the 5, 10, 27, and 133-dimensional representations of $SL(5)$, $SO(5,5)$, E_6 and E_7 respectively. For the first few entries many contributions vanish as one has too many antisymmetrised indices. To find the 3875 dimensional representation of E_8 one must go further in the analysis.

The factor of e^ρ associated with the term at n_c is

$$e^{-\frac{1}{\sqrt{2}}(\frac{\vec{\nu}\cdot\vec{\lambda}^{n+1}}{x} - n_c x)\rho} = e^{-\frac{2\alpha}{n}(n-2n_c(8-n))\rho} \ . \tag{6.57}$$

The corresponding field strengths carry three d dimensional spacetime indices and $2n_c$ internal indices and so we find in the dimensionally reduced theory a factor of

$$e^{-\rho(3\alpha+(2n_c)\beta)} = e^{-\frac{\alpha\rho}{n}(3n-2n_c(8-n))} \ . \tag{6.58}$$

Consequently for every rank three field strength we find an additional factor of $e^{-\alpha\rho}$ in the dimensionally reduced theory. The same conclusion holds for the rank four field strengths.

Since one finds the same additional factor no matter what field strength one considers the above can be summarised as follows, for every derivative we find an additional factor of $e^{-\alpha\rho}$ in the dimensionally reduced theory. One also finds in the dimensionally reduced theory a $e^{-\hat{s}\phi}$ coming from the ten dimensional $SL(2)$ automorphic form. Consequently, the excess in the dimensionally reduced theory compared to that found in the E_{n+1} formulation of equation (6.57), but not taking into account the contribution of the E_{n+1} automorphic form in d dimensions in the latter formulation, is given by

$$e^{-(l_T-2)\alpha\rho - \hat{s}\phi} \ , \tag{6.59}$$

where $l_T = \hat{l}_R + \hat{l}_1 + \hat{l}'_1 + \hat{l}_3 + \hat{l}'_3 + \hat{l}_5$. The -2 part arises from the $\det \hat{e}$. This excess can only come from the E_{n+1} automorphic form. Demanding that all the weights arising from dimensional reduction of the ten dimensional theory appear in the E_{n+1} formulation in d dimensions we conclude that

$$\vec{\Lambda}_\Phi = \left(\frac{\hat{s}}{\sqrt{2}}, \alpha\frac{(l_T-2)}{\sqrt{2}}, \underline{0}\right) = \left(\frac{l_T-2}{4} + (l_{RR}-2g)\right)\vec{\Lambda}^{n+1} + \left(\frac{2g-l_{RR}}{2}\right)\vec{\Lambda}^n \ , \tag{6.60}$$

where $l_{RR} = \hat{l}'_1 + \hat{l}'_3 + \hat{l}_5$ counts the number of R-R fields.

Let us consider higher derivative terms constructed only out of NS-NS fields, so that $l_{RR} = 0$. Suppose also that we look at terms which have a tree level, $g = 0$, contribution in ten-dimensions. In this case we find the automorphic form in d dimensions has the leading order behaviour $\Phi_E \sim e^{-\sqrt{2}\frac{(l_T-2)}{4}\vec{\Lambda}^{n+1}}$. This strongly suggests that it is built from the E_{n+1} representation with highest weight $\vec{\Lambda}^{n+1}$. This is the representation that the string charges of the d dimensional theory belong to.

The $SL(2,\mathbb{Z})$ Eisenstein automorphic form in ten dimensions contains two perturbative terms with dilaton dependence $e^{-s\phi}$ and $e^{(s-1)\phi}$. If the first term possesses a value of s that leads to a tree level contribution then the second term leads to a genus $g = s - 1/2$ contribution. Above we considered the effect of dimensionally reducing the tree level contribution, but one can also consider the second contribution. One finds, substituting $g = s - 1/2$ into (6.60), that the weight vector is

$$\vec{\Lambda}_\Phi = (1-s)\vec{\Lambda}^{n+1} + (s-1/2)\vec{\Lambda}^n = s\vec{\Lambda}^{n+1} - (s-1/2)\vec{\alpha}_{n+1} \ . \tag{6.61}$$

However the first two terms in the perturbative contribution of the Eisenstein-like E_{n+1} automorphic form in d dimensions constructed using the $\vec{\lambda}^{n+1}$ representation are of the generic form [40]

$$\Phi_E \sim E_1 e^{-\sqrt{2}s\vec{\Lambda}^{n+1}} + E_2 e^{-\sqrt{2}(s\vec{\Lambda}^{n+1} - (s-1/2)\vec{\alpha}_{n+1})} \ , \tag{6.62}$$

where E_1 and E_2 are constants. It is pleasing to see that the second term of the automorphic form in ten dimensions leads to the correct second term in the E_{n+1} automorphic form in d dimensions.

6.2 M-Theory

Let us now perform a similar analysis for the dimensional reduction of higher derivative terms of M-theory. Note that to compare with the previous section one must make the substitution $n \to n+1$. In addition the values of α and x in this section are different to those in section 6.1. The Bosonic field content of M-theory consists of the graviton with curvature $\hat{R}$ and a three form gauge field $\hat{A}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ out of which the four-form field strength $\hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$ is constructed. At lowest order in derivatives the effective action may be written

$$\int d^{11}x \det \hat{e} \left(\hat{R} - \frac{1}{2\cdot 4!}\hat{F}^2 + \dots \right) \ . \tag{6.63}$$

where the ellipsis denote Fermion terms as well as a Chern-Simons-type term for $\hat{A}$. A generic higher derivative correction in the $d = 11$ effective action of M-theory may be written,

$$\int d^{11}x \det \hat{e} (\partial)^{\hat{l}_0} \hat{R}^{\frac{\hat{l}_1}{2}} \hat{F}^{\hat{l}_4}. \tag{6.64}$$

M-theory, dimensionally reduced on an n-torus, possesses an E_n symmetry in $d = 11-n$ dimensions and shares the same manifest $SL(n)$ symmetry through the non-linearly realised field strengths and the Cartan forms in d dimensions as the type IIB theory. However, no dilatonic scalar is present in $d = 11$ dimensions. Upon dimensional reduction, a higher derivative term will pick up a dependence on the n diagonal components of the metric on the n-torus ρ and $\underline{\phi}$. We observe that the higher derivative terms in the dimensionally reduced formulation carry a factor of $e^{\sqrt{2}\vec{w}\cdot\vec{\phi}}$ where the n vectors $\vec{w}$ and $\vec{\phi}$ recording the dilatonic scalar field content and their associated weights are defined as

$$\begin{aligned}\vec{\phi} &= \left(\rho, \underline{\phi}\right), \\ \vec{w} &= \left(\kappa, \underline{w}\right).\end{aligned} \tag{6.65}$$

The general term in the E_n formulation in d dimensions is a polynomial in the non-linearly realised field strengths $\mathcal{F}$, Cartan forms P and curvature R multiplied by an automorphic form Φ_E constructed out of some representation of E_n

$$\int d^d x \det e \partial^{l_0} R^{\frac{l_R}{2}} P^{l_1}_{E\mu_1} (\mathcal{F}_{E\mu_1\mu_2})^{l_2} (\mathcal{F}_{E\mu_1\mu_2\mu_3})^{l_3} \Phi_E \ldots . \tag{6.66}$$

We will again determine the representation out of which the E_n automorphic form is constructed in d dimensions by comparing the dimensionally reduced formulation, with manifest $SL(n)$ symmetry, to that of the E_n formulation. The Dynkin diagram for M-theory is given in figure 11. Note that here we use a different labeling for the nodes and hence the roots and weights are also labeled differently than in section 6.1.

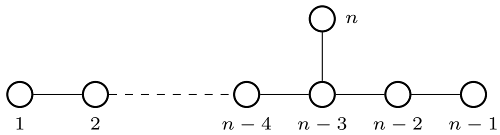

Figure 11: Dynkin diagram for E_n with M-theory labeling

The simple roots of E_n may be written as

$$\begin{aligned}\vec{\alpha}_i &= (0, \underline{\alpha}_i), \quad i = 1, ..., n-1, \\ \vec{\alpha}_n &= (x, \underline{0}) - \vec{\mu},\end{aligned} \tag{6.67}$$

where $\vec{\mu} = \left(0, \underline{\lambda}^{n-3}\right)$. The variable x associated with the ρ factor of the deleted node n is evaluated via the inner products between the simple roots of E_n, given by the corresponding Cartan matrix, one finds

$$x = \sqrt{\frac{9-n}{n}} = \left(3\sqrt{2}\alpha\right)^{-1} . \tag{6.68}$$

The fundamental weights of E_n, dual to the simple roots $\underline{\alpha}_i$, are

$$\begin{aligned}\vec{\Lambda}^i &= \left(\frac{1}{x}\underline{\lambda}^i \cdot \underline{\lambda}^{n-3}, \underline{\lambda}^i\right), \\ \vec{\Lambda}^n &= \left(\frac{1}{x}, \underline{0}\right).\end{aligned} \tag{6.69}$$

One may write any root of E_n as

$$\vec{\alpha} = n_c \vec{\alpha}_n + \sum_{i=1}^{n-1} n_i \vec{\alpha}_i = n_c \left(x, \underline{0}\right) - \vec{\Lambda} \ , \tag{6.70}$$

where $\vec{\Lambda} = n_c \vec{\nu} - \sum_{i=1}^{n-1} n_i \vec{\alpha}_i$. As in the IIB theory, if a representation of $SL(n)$ is present at some level n_c in the adjoint representation of E_n, then its highest weight may be written as $\vec{\Lambda}$ for some combination of the integers n_c and n_i. Level $n_c = 0$ contains the adjoint representation of $SL(n)$. The highest weight representations of $SL(n)$ at higher levels are

$$\begin{array}{ccc} n_c = 1 & n_c = 2 & n_c = 3 \\ \underline{\lambda}^3 & \underline{\lambda}^6 & \underline{\lambda}^1 + \underline{\lambda}^{n-8} \ . \end{array} \tag{6.71}$$

So the weights in the lower levels of the adjoint representation of E_n are

$$\left(0, \left[\underline{\alpha}_1 + ... + \underline{\alpha}_{n-1}\right]\right) \ , \left(x, \left[\underline{\lambda}^3\right]\right) \ , \left(2x, \left[\underline{\lambda}^6\right]\right) \ , \left(3x, \left[\underline{\lambda}^1 + \underline{\lambda}^{n-8}\right]\right) \ . \tag{6.72}$$

The decomposition of the Cartan form $\mathcal{S}_E$, at a given level n_c is found by examining the weights. At level $n_c = 0$ the Cartan form $\mathcal{S}_E$ contains the Cartan form of $SL(n)$ at higher levels the Cartan form $\mathcal{S}_E$ decomposes as follows

$$\begin{array}{ccc} n_c = 1 & n_c = 2 & n_c = 3 \\ S_{SL(n) i_1 i_2 i_3} & S_{SL(n) i_1 \ldots i_6} & S_{SL(n) j, i_1 \ldots i_8} \ . \end{array} \tag{6.73}$$

The Cartan form contains the factor $e^{\frac{1}{\sqrt{2}} \vec{\phi} . \vec{\alpha}}$, so at level n_c we find a factor of

$$e^{\frac{1}{\sqrt{2}}((n_c x)\rho)} = e^{(3n_c)\alpha\rho\left(\frac{9-n}{n}\right)}. \tag{6.74}$$

With the natural ordering on the levels, we find the maximum level that contributes is $n_c = 1$ for $n = 3, 4$, $n_c = 2$ for $n = 5, 6$ and $n_c = 3$ for $n = 7$. The $SL(n)$ Cartan forms $S_{SL(n)}$ originate from the four-form field strength $\hat{G}_{\mu \bar{i}_1 \bar{i}_2 \bar{i}_3}$ at level $n_c = 1$, the dual of the four-form field strength at level $n_c = 2$ and the graviphoton at level $n_c = 3$. These Cartan forms of $SL(n)$, arising upon dimensional reduction, carry one d dimensional spacetime index and $(3n_c)$ internal indices. Therefore, each Cartan form of $SL(n)$, at level n_c, is multiplied by the factor

$$e^{-\rho(\alpha + (3n_c)\beta)} = e^{(3n_c)\alpha\rho\left(\frac{9-n}{n}\right)} e^{-\alpha\rho}. \tag{6.75}$$

The two-form field strengths lie in the representation of E_n with highest weight $\vec{\Lambda}^1$. Decomposing the $\vec{\Lambda}^1$ of E_n into representations of $SL(n)$ level by level, we find

$$\begin{array}{cccc} n_c=0 & n_c=1 & n_c=2 & n_c=3 \\ \underline{\lambda}^1 & \underline{\lambda}^{n-2} & \underline{\lambda}^{n-5} & \underline{\lambda}^{n-1}+\underline{\lambda}^{n-7}. \end{array} \tag{6.76}$$

Therefore, for $n \leq 7$, the weights of the $\vec{\Lambda}^1$ representation are

$$\left(\frac{3}{nx},[\underline{\lambda}^1]\right),\left(\frac{3}{nx}-x,[\underline{\lambda}^{n-2}]\right),\left(\frac{3}{nx}-2x,[\underline{\lambda}^{n-5}]\right),\left(\frac{3}{nx}-3x,[\underline{\lambda}^{n-1}+\underline{\lambda}^{n-7}]\right). \tag{6.77}$$

From the weights, we see that the corresponding two-form field strengths, at each level, are

$$\begin{array}{cccc} n_c=0 & n_c=1 & n_c=2 & n_c=3 \\ \mathcal{F}^i_{\mu_1\mu_2} & \mathcal{F}_{\mu_1\mu_2 i_1 i_2} & \mathcal{F}_{\mu_1\mu_2 i_1\ldots i_5} & \mathcal{F}_{\mu_1\mu_2 i,j_1\ldots j_7}. \end{array} \tag{6.78}$$

The two-form field strengths appear in $d \geq 4$ dimensions. In $d = 11-n$ dimensions one finds that all two-form field strengths, with associated level n_c, satisfying the constraint $n \leq 3n_c - 1$ will be present. We see that the two-form field strength at level $n_c = 0$ arises through the dimensional reduction of the metric and four-form at levels $n_c = 0$, $n_c = 1$ respectively. The two remaining levels in the decomposition of the $\vec{\Lambda}^1$ are associated with the duals of the four-form field strength and the graviphoton at $n_c = 2$ and $n_c = 3$ respectively. Since the two form field strengths in the E_n lie in some representation of $SL(n)$ at level n_c in the decomposition of $\vec{\Lambda}^1$ they carry a multiplicative factor of

$$e^{\frac{1}{\sqrt{2}}\left(\left(-\frac{3}{nx}+n_c x\right)\rho\right)} = e^{-\frac{9}{n}-n_c\alpha\rho\left(\frac{9-n}{n}\right)}. \tag{6.79}$$

If we compare the multiplicative factor found through the decomposition of the $\vec{\Lambda}^1$ in the E_n formulation to the corresponding factor arising in the dimensionally reduced formulation, where the two-form field strengths carry two d dimensional indices and $3n_c - 1$ internal indices, and so appear multiplied by the factor

$$e^{-\rho(2\alpha+(3n_c-1)\beta)} = e^{-\alpha\rho}e^{-\frac{9}{n}-n_c\alpha\rho\left(\frac{9-n}{n}\right)}, \tag{6.80}$$

we find that the two-form field strengths in the dimensionally reduced M-theory formulation carry a surplus factor of $e^{-\alpha\rho}$. In the E_n formulation the three form field strengths lie in the representation with highest weight $\vec{\Lambda}^{n-1}$. One finds that the $\vec{\Lambda}^{n+1}$ representation of E_n decomposes, in the following way for $n \leq 5$

$$\begin{array}{cc} n_c=0 & n_c=1 \\ \underline{\lambda}^{n-1} & \underline{\lambda}^{n-4} \end{array}. \tag{6.81}$$

We observe that, for $n \leq 5$, the weights in the $\vec{\Lambda}^{n-1}$ representation of E_n are

$$\left(\left(\frac{n-3}{nx}\right),\ \left[\underline{\lambda}^{n-1}\right]\right),\left(\frac{(n-3)}{nx}-x,\ \left[\underline{\lambda}^{n-4}\right]\right). \tag{6.82}$$

The three form field strengths, at level n_c, are

$$\begin{array}{cc} n_c=0 & n_c=1 \\ \mathcal{F}_{\mu_1\mu_2\mu_3 i} & \mathcal{F}_{\mu_1\mu_2\mu_3 i_1\ldots i_4}. \end{array} \tag{6.83}$$

The three form field strength occurring in the decomposition of the Λ^{n-1} at level $n_c = 0$ arises from the dimensional reduction of the four-form field strength, while the other, at level $n_c = 1$ is associated with the dual of the dimensionally reduced four-form field strength. The three form field strengths at levels $n_c = 0, 1$ appear in $d = 6, 7$ dimensions, in $d = 8$ only the $n_c = 0$ three form field strength is present. The decomposition of the $\vec{\Lambda}^{n-1}$ of E_n, at level n_c, is multiplied by a factor of

$$e^{-\frac{1}{\sqrt{2}}\left(\frac{(n-3)}{nx}-n_c x\right)\rho} = e^{\left(\left(-3+\frac{9}{n}\right)+n_c\left(\frac{9-n}{n}\right)\right)\alpha\rho}. \tag{6.84}$$

The three form field strengths in the dimensionally reduced formulation come with three spacetime indices and $(3n_c + 1)$ internal indices, therefore they carry a factor of

$$e^{-\rho(3\alpha+(3n_c+1)\beta)} = e^{-\alpha\rho}e^{\left(\left(-3+\frac{9}{n}\right)+n_c\left(\frac{9-n}{n}\right)\right)\alpha\rho}. \tag{6.85}$$

In $d = 11 - n$ dimensions, the Cartan forms, field strengths and curvatures lying in the E_n representation may be constructed out of the dimensionally reduced Cartan forms, field strengths and curvatures with manifest $SL(n)$ symmetry. For example, the two form field strengths in $d = 7$ dimensions lie in the **10** of E_4, which may be constructed out of the two-form field strengths arising from dimensional reduction to $d = 7$. Namely, the graviphotons lying in the **4** and the dimensionally reduced four-form field strength $\hat{G}_{\mu_1\mu_2\bar{i}_1\bar{i}_2}$ lying in the $\underline{\mathbf{6}}$ of $SL(4)$. However, each of the dimensionally reduced terms carry an additional factor of $e^{-\alpha\rho}$. Therefore, any product of Cartan forms, field strengths and curvatures, in the E_n formulation, reconstructed using the appropriate dimensionally reduced terms, will be multiplied by a surplus factor of

$$e^{-(l_T-2)\alpha\rho}, \tag{6.86}$$

where l_T is the total number of derivatives in the product. This factor must be attributed to the automorphic form in the E_n formulation. To leading order, we may write the automorphic form in the E_n formulation as $\Phi_{E_n} \sim e^{-\sqrt{2}\vec{\Lambda}_\Phi\cdot\vec{\phi}}$. Thus, one finds

$$\vec{\Lambda}_\Phi = \left(\alpha\left(\frac{l^T-2}{\sqrt{2}}\right),\underline{0}\right) = \left(\frac{l_T-2}{6}\right)\vec{\Lambda}^n. \tag{6.87}$$

6.3 Conclusion

We have dimensionally reduced the higher derivative terms of ten dimensional IIB theory and deduced the weight vectors that are associated with the Cartan subalgebra fields of the E_{n+1} symmetry. Most of these weights are accounted for once the d-dimensional theory is expressed in terms of E_{n+1} covariant building blocks involving the Riemann tensor, field strengths and derivatives of the scalars. However, we also found that there was always a remaining weight. This implies that polynomials constructed only out of the field strengths are not consistent with U-duality in the lower dimension. On the other hand these additional weights can be accounted for in the d dimensional theory if they are attributed to an E_{n+1} automorphic form. In this way we obtained constraints on the automorphic forms that occur in d-dimensions.

Carrying out this procedure we have found that the dimensional reduction of the IIB higher derivative corrections implies that such terms in d dimensions should contain an automorphic form involving the weight $\vec{\Lambda}^{n+1}$, using the labeling of the Dynkin diagram of Figure 9. It is natural to think of this as the highest weight of the representation used to construct the automorphic form. This applies to all terms in a given dimension, although this does not mean that the same automorphic form appears for all terms. For terms that only contain the Riemann tensor and scalars the leading order weight can be readily deduced by counting the number of inverse metrics required, however for more general terms we needed to perform a detailed group theory analysis.

As the constraints we find arise from considering the ten dimensional theory we are in effect considering terms that survive the decompactification from d dimensions, that is $\rho \to -\infty$. We have focused particularly on the terms that arise at tree level in ten dimensions. However we also saw that the next-to-leading order contribution in ten-dimensions correctly matched that of the d-dimensional automorphic form if the $\vec{\Lambda}^{n+1}$ representation is used for the case of Einstein-like automorphic forms.

This result is in agreement with the results [15, 23, 24, 27, 31–48] found so far for terms with low numbers of spacetime derivatives in that the automorphic forms studied for these terms are constructed from the $\vec{\Lambda}^{n+1}$ representation. It is also natural in that the string charges belong to the $\vec{\Lambda}^{n+1}$ multiplet and the discrete E_{n+1} group acts naturally on these objects.

We also performed a similar calculation from the viewpoint of eleven-dimensional M-theory. We found that the automorphic forms should contain the weight $\vec{\Lambda}^{n-1}$, using the type IIB labeling of the E_{n+1} Dynkin diagram of Figure 9. This is also natural as membrane charges belong to the $\vec{\Lambda}^{n-1}$ representation. The automorphic forms contain combinations of weights and one would have to find the combination of weights predicted from the M-theory viewpoint in the automorphic form constructed from the representation with highest weight $\vec{\Lambda}^{n+1}$ that it used in the type IIB theory. In this way the M-theory analysis places a non-trivial constraint on the automorphic forms. In the next chapter we will perform a similar calculation for the more complicated case of type IIA string theory and put forward a conjecture that reconciles the seemingly different weights of the E_{n+1} automorphic form found upon dimensional reduction of M-theory and type IIB string theory, in conjunction with the result found for type IIA string theory.

7 Constraints on Type IIA Automorphic Forms

In chapter 6 we found that the dimensional reduction of higher derivative terms in the type IIB string theory and M-theory effective actions on an n torus leads to constraints in the form of particular weights of $E_{n+1}(\mathbb{R})$ that the type IIB and M-theory $E_{n+1}(\mathbb{Z})$ automorphic forms found in $d = 10 - n$ dimensions must contain in the large volume limit of the torus. However, attempts to derive similar constraints for the type IIA theory is complicated by the fact that the type IIA dilaton mixes with the torus moduli upon dimensional reduction. In this chapter we use the E_{11} formulation of type IIA supergravity to resolve this issue and perform an analogous calculation to that performed in chapter 6 for type IIB string theory and M-theory. In particular we will consider the dimensional reduction of the higher derivative string corrections of the IIA theory on an n-dimensional torus to $d = 10 - n$ dimensions. We will compare these with the higher derivative corrections that arise in the d dimensional theory assuming that the theory is invariant under an $E_{n+1}(\mathbb{Z})$ symmetry and so possess a corresponding automorphic form built from a representation of E_{n+1}, where the E_{n+1} Dynkin diagram with type IIA string theory labeling is given in figure 12. This comparison allows us to also place constraints on the representation used for any higher derivative correction.

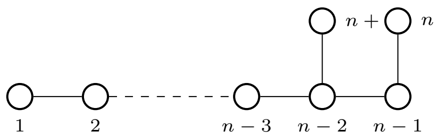

Figure 12: Dynkin diagram for E_{n+1} in type IIA labeling

We show that the higher derivative terms can be written as powers of the E_{n+1} covariant field strengths, E_{n+1} Cartan forms and d dimensional scalar curvatures along with additional factors of the dilaton and volume given by $e^{\sqrt{2}\vec{w}.\vec{\phi}}$, where $\vec{w}$ is an n dimensional vector proportional to an element of the weight lattice of E_{n+1}. For terms that arise at tree level in string perturbation theory in ten dimensions we find $\vec{w} = s\vec{\Lambda}_n$, where $s = (l_T - 2)/4$ with l_T counting the number of derivatives and $\vec{\Lambda}_n$ the fundamental weight dual to $\vec{\alpha}_n$ (see figure 12). The observation of chapter six and reference [28] is that these additional factors must come from an automorphic form and therefore we conclude that the automorphic form which multiplies a given higher derivative term involves the weight $\vec{\Lambda}_n$. For Eisenstein-like automorphic forms the leading order behaviour is given by $e^{-\sqrt{2}s\vec{\phi}.\vec{\Lambda}^H}$, where $\vec{\Lambda}^H$ is the highest weight of the representation used to construct the automorphic form. Thus our results suggest that the higher derivative terms always include an automorphic form built from a representation with highest weight $\vec{\Lambda}^H$. It is important to note that we are considering a particular limit and other representations could also appear but be subdominant in this limit.

In order to carry out the comparison we need to identify the fields that arise in the dimensional reduction from ten dimensions with the fields that occur in the formulation of the d-dimensional theory in which the $E_{n+1}(\mathbb{Z})$ symmetry is manifest, and in particular the scalar fields from which the automorphic form is constructed. This identification can be carried out in the context of the supergravity theories. The most obvious technique is to explicitly carry out the dimensional reduction and reformulate the theory with the manifest $E_{n+1}(\mathbb{Z})$ symmetry, but this is rather lengthy and complicated involving dualisations and other subtleties. In this chapter we will use the E_{11} formulation of the IIA theory.

In the E_{11} formulation the fields of the IIA theory are in one to one correspondence with the generators of the Borel subalgebra of E_{11}. As the E_{11} algebra contains in an obvious way the E_{n+1} algebra, the correspondence between the scalar fields that appear in the non-linear realisation of E_{n+1} and the E_{11} generators is easily found. However, the correspondence between the E_{11} generators and the fields usually used to formulate the IIA supergravity theory is known from the formulation of this theory as a non-linear realisation at lowest levels in E_{11}. Thus one finds the desired relation between the fields of the IIA theory and the scalars fields associated with E_{n+1} in a simple way. We note that this use of E_{11} does not depend on the conjecture that E_{11} is an underlying symmetry of the theory of strings and branes.

7.1 The Dimensional Reduction

The bosonic field content of type IIA supergravity in ten dimensions consists of a scalar (the type IIA dilaton ϕ), a NS-NS three form field strength $\hat{F}_{\mu_1\mu_2\mu_3}$ constructed from the NS-NS two-form gauge field $A_{\mu_1\mu_2}$, in addition to two R-R form field strengths $\hat{F}_{\mu_1\mu_2}$, $\hat{F}_{\mu_1\mu_2\mu_3\mu_4}$ constructed from the R-R gauge fields A_{μ_1} and $A_{\mu_1\mu_2\mu_3}$. In Einstein frame, the bosonic part of the type IIA supergravity action is given by,

$$\begin{aligned} S_{IIA} = \frac{1}{2\kappa_{10}^2}\int d^{10}x det(\tilde{e})\Bigg(\tilde{R} - \frac{1}{2\cdot 4!}e^{\frac{1}{2}\phi}\tilde{F}_{\mu_1\mu_2\mu_3\mu_4}\tilde{F}^{\mu_1\mu_2\mu_3\mu_4} - \frac{1}{2\cdot 3!}e^{-\phi}\tilde{F}_{\mu_1\mu_2\mu_3}\tilde{F}^{\mu_1\mu_2\mu_3} \\ -\frac{1}{2\cdot 2!}e^{\frac{3}{2}\phi}\tilde{F}_{\mu_1\mu_2}\tilde{F}^{\mu_1\mu_2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\Bigg), \end{aligned} \tag{7.1}$$

where κ_{10} is a constant related to the Newton constant in ten dimensions and

$$\tilde{F}_{\mu_1\mu_2} = 2\partial_{[\mu_1}\tilde{A}_{\mu_2]}, \tag{7.2}$$

$$\tilde{F}_{\mu_1\mu_2\mu_3} = 3\partial_{[\mu_1}\tilde{A}_{\mu_2\mu_3]}, \tag{7.3}$$

$$\tilde{F}_{\mu_1\mu_2\mu_3\mu_4} = 4\left(\partial_{[\mu_1}\tilde{A}_{\mu_2\mu_3\mu_4]} + \tilde{A}_{[\mu_1}\tilde{F}_{\mu_2\mu_3\mu_4]}\right). \tag{7.4}$$

We have suppressed the Chern-Simons term since it will not play a part in our analysis. The type IIA supergravity action possesses a $GL(1,R)$ symmetry, that manifests itself through a shift of the type IIA dilaton and a scaling of the other fields. One can introduce the following combinations of the fields strengths and dilaton that are inert under $GL(1,R)$ transformations

$$\tilde{\mathcal{F}}_{\mu_1\mu_2\mu_3\mu_4} = e^{\frac{1}{4}\phi}\tilde{F}_{\mu_1\mu_2\mu_3\mu_4}, \tag{7.5}$$

$$\tilde{\mathcal{F}}_{\mu_1\mu_2\mu_3} = e^{-\frac{1}{2}\phi}\tilde{F}_{\mu_1\mu_2\mu_3}, \tag{7.6}$$

$$\tilde{\mathcal{F}}_{\mu_1\mu_2} = e^{\frac{3}{4}\phi}\tilde{F}_{\mu_1\mu_2}. \tag{7.7}$$

In fact these are just the non-linear representations of $GL(1,R)$ constructed from the linear representations in the usual way. They are inert, as the local subalgebra is the identity group. Rewriting the action with these objects effectively absorbs the dilaton factors multiplying the field strengths in (7.1), the action then becomes

$$\begin{aligned} S_{IIA} = \frac{1}{2\kappa_{10}^2}\int d^{10}x det(\tilde{e})\Big(\tilde{R} - \frac{1}{2\cdot 4!}\tilde{\mathcal{F}}_{\mu_1\mu_2\mu_3\mu_4}\tilde{\mathcal{F}}^{\mu_1\mu_2\mu_3\mu_4} - \frac{1}{2\cdot 3!}\tilde{\mathcal{F}}_{\mu_1\mu_2\mu_3}\tilde{\mathcal{F}}^{\mu_1\mu_2\mu_3} \\ -\frac{1}{2\cdot 2!}\tilde{\mathcal{F}}_{\mu_1\mu_2}\tilde{\mathcal{F}}^{\mu_1\mu_2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\Big). \end{aligned} \tag{7.8}$$

We are interested in the dimensional reduction of a generic term higher derivative term which may be written as

$$\int d^{10}x det\tilde{e}\partial^{l_0}\tilde{R}^{\frac{\tilde{l}_R}{2}}(\tilde{P}_{\mu_1})^{\tilde{l}_1}(\tilde{\mathcal{F}}_{\mu_1\mu_2})^{\tilde{l}_2}\tilde{\mathcal{F}}_{\mu_1\mu_2\mu_3})^{\tilde{l}_3}(\tilde{\mathcal{F}}_{\mu_1\mu_2\mu_3\mu_4})^{\tilde{l}_4}\Phi_{\tilde{s}}, \tag{7.9}$$

where $\Phi_{\tilde{s}}$ is a function of ϕ that is of the form $\Phi_{\hat{s}} = e^{-\tilde{s}\phi}$. Dimensional reduction on the n torus to a theory in $d = 10 - n$ dimensions is achieved using our standard metric compactification ansatz

$$d\tilde{s}_{10}^2 = e^{2\alpha\rho}g_{\mu\nu}dx^\mu dx^\nu + e^{2\beta\rho}G_{ij}\left(dx^i + A^i_\mu dx^\mu\right)\left(dx^j + A^j_\nu dx^\nu\right), \tag{7.10}$$

where the background and internal metrics are denoted $g_{\mu\nu}$ and G_{ij} respectively, with the internal metric satisfying $det(G) = 1$ and the constants α and β are given by equations 3.12 and 3.13. The internal vielbein is given by $e_i{}^{\underline{k}}e_j{}^{\underline{l}}\delta_{\underline{kl}} = G_{ij}$ and it also has $\det(e) = 1$. Tangent internal indices possess an underline as shown. The gauge fields are dimensionally reduced using the method described in chapter three and a notation is adopted where greek letters denote $d = 10-n$ dimensional spacetime indices while lower case Roman letters are internal $SL(n)$ indices.

We will be interested in the dependence of the above ten dimensional higher derivative correction in string frame. The transition from Einstein frame to string frame is given by $\tilde{e} = e^{-\frac{\phi}{4}}\tilde{e}_s$. The term in (7.9) then leads to the factor

$$e^{\frac{\phi}{4}\left(\tilde{l}_0+\tilde{l}_R+\tilde{l}_1+5\tilde{l}_2+\tilde{l}_3+5\tilde{l}_4-10-4\tilde{s}\right)}. \tag{7.11}$$

At order g in perturbation theory we have the contribution $e^{\phi(2g-2)}$ and so for a perturbative contribution we find

$$\tilde{s} = \frac{1}{4}\left(\tilde{l}_0 + \tilde{l}_R + \tilde{l}_1 + 5\tilde{l}_2 + \tilde{l}_3 + 5\tilde{l}_4 - 2 - 8g\right). \tag{7.12}$$

The dimensionally reduced theory will contain field strengths of the form $F_{\mu_1\ldots\mu_p i_1\ldots i_k}$ where the internal indices $i_1,\ldots,i_k$ are world volume indices. The theory in d dimensions possess the $GL(1,\mathbb{R})$ symmetry of the IIA theory, but in addition has an $SL(n,\mathbb{Z})$ symmetry corresponding to diffeomorphisms that are preserved by the torus. We can convert the internal world indices to tangent frame indices using the inverse internal vielbein. However, as explained in reference [28] page 5, the internal vielbein is just the group element of the non-linear realisation of $SL(n)$ with local subgroup $SO(n)$ in the vector representation. Carrying this out on the $GL(1,\mathbb{R})$ inert objects of equation (7.7) we find $\mathcal{F}_{j_1\ldots j_k}$, with any space-time indices suppressed, which converts to tangent space as follows

$$\mathcal{F}_{SL(n)\otimes GL(1)\ \underline{i}_1\ldots\underline{i}_k} = (e^{-1})_{\underline{i}_1}{}^{j_1}\ldots(e^{-1})_{\underline{i}_k}{}^{j_k}F_{j_1\ldots j_k}, \tag{7.13}$$

where $e_i{}^{\underline{k}}$ is the vielbein on the torus and $F_{j_1\ldots j_k}$ transforms in the linear representation of $SL(n)$ with highest weight λ_k. Thus $\mathcal{F}_{SL(n)\otimes GL(1)\ \underline{i}_1\ldots\underline{i}_k}$ transforms as a non-linear representation constructed from a linear representation in the standard way. This is consistent with the action of the tangent space group on the vielbein. If we use tangent space internal indices then the $SL(n)$ symmetry will be essentially manifest as long as we construct $SO(n)$ invariants.

If we denote the part of the group element of the non-linear realisation of $SL(n)$ with local subgroup $SO(n)$ which contains the Cartan generators $\underline{H}$ of $SL(n)$ by $g_{SL(n)} = e^{\underline{H}\cdot\underline{\phi}_k}$ then the dimensionally reduced field strength $\mathcal{F}_{SL(n)\otimes GL(1)\ \underline{i}_1\ldots\underline{i}_k}$ carries a factor of $e^{\underline{\phi}\cdot[\lambda_k]}$ where λ_k is the $SL(n)$ representation with highest weight λ_k and $[\lambda_k]$ is a weight in this representation. When written in terms of the field strengths using equation (7.7) we also find exponential factors involving the dilaton. One can incorporate these automatically by considering the group $SL(n)\otimes GL(1)$ with the group element $g_{SL(n)\otimes GL(1)} = e^{\phi R}e^{\underline{H}\cdot\underline{\phi}_k}$ where R is the generator of $GL(1,\mathbb{R})$.

The dimensional reduction of terms containing field strengths in the IIA higher derivative theory will lead to terms containing the object $\mathcal{F}_{SL(n)\otimes GL(1)}$ of equation (7.11) multiplied by exponentials of the field ρ which arises from dimensional reduction using the ansatz of equation (7.10). The field strength of equation (7.11) has a ρ factor given by $e^{-k\beta\rho}$. If we were to convert the space-time world indices to tangent indices then we would also acquire a factor $e^{-p\alpha\rho}$ if we have p space-time indices.

The derivatives of the scalars, including the dilaton, are contained in the Cartan forms of $SL(n) \otimes GL(1)$ and in particular the parts which change by a minus sign under the action of the Cartan involution, and are denoted by $S_{SL(n)\otimes GL(1)}$. The action of the Cartan involution on $SL(n)$ generators is such as to lead to $SO(n)$ being the invariant group and on the generator R it acts with a minus sign.

After dimensional reduction, the IIA theory including the higher derivative terms can be expressed in terms of the scalar curvature R, which is an $SL(n)$ singlet, the $S_{SL(n)\otimes GL(1,\mathbb{R})}$ part of the Cartan forms of $SL(n) \otimes GL(1,\mathbb{R})$ and the field strengths $F_{SL(n)\otimes GL(1,\mathbb{R})\ \mu_1\cdots\mu_p\underline{i}_1\cdots\underline{i}_k}$ which transform as non-linear representations of $SL(n) \otimes GL(1,\mathbb{R})$.

7.2 The E_{n+1} Formulation in d Dimensions

As is well known, the type II supergravity theories in d dimensions possess an E_{n+1} symmetry. Their actions are bilinear in the space-time derivatives and include the Riemann curvature, and squares of the field strength and the derivatives of the scalars. The metric, in Einstein frame, transforms as a singlet of E_{n+1} and therefore the Riemann curvature is invariant under E_{n+1} transformations. The scalars belong to the non-linear realisation of E_{n+1} with a local subgroup H_{n+1} which is the maximal compact subgroup. The latter is just the the Cartan involution invariant subgroup. This means that the scalars are contained in a group element $g_{E_{n+1}} \in E_{n+1}$ which transforms as $g_{E_{n+1}} \to g_0 g_{E_{n+1}}$ where $g_0 \in E_{n+1}$ is independent of space-time and also $g_{E_{n+1}} \to g_{E_{n+1}} h$ where $h \in H_{n+1}$ and is an arbitrary function of space-time. We can write the Cartan subalgebra part of the group element as $g_{E_{n+1}} = e^{\vec{\phi}.\vec{H}}$ where $\vec{H}$ are the $n+1$ Cartan subalgebra generators of E_{n+1}, which we have written as a vector. The corresponding scalar fields are written as the vector $\vec{\phi}$.

The non-linear realisation essentially specifies how the scalars appear in the action. In particular, the derivatives of the scalars occur as Cartan forms of E_{n+1} in the coset directions. In terms of our group element $g_{E_{n+1}}$, the Cartan forms which are given by $g_{E_{n+1}}^{-1} dg_{E_{n+1}}$ in the coset directions, are denoted $\mathcal{S}_{E_{n+1}}$. When evaluated they contain the roots $\vec{\alpha}$ and so are given by terms of the form $e^{\vec{\phi}.\vec{\alpha}}$ where $\vec{\alpha}$ are the roots of E_{n+1}.

The gauge fields occur in the field strengths F that transform as linear representations of E_{n+1}. However, we can convert a linear representation of E_{n+1} into a non-linear representation using a group element $g_{E_{n+1}}^{-1}$. Explicitly, the non-linear representation $|\mathcal{F}\rangle$ constructed from a linearly realised field strength $|F\rangle$ is given by

$$|\mathcal{F}_{E_{n+1}}\rangle = L(g_{E_{n+1}}^{-1})|F\rangle, \tag{7.14}$$

where $L((g_{E_{n+1}}(\xi))^{-1})$ is the representation with highest weight Λ_k. From equation (7.14) we find that the non-linearly realised field strength $|\mathcal{F}_{E_{n+1}}\rangle$ contains a dependence on the scalars $\vec{\phi}$ which is given by $e^{\vec{\phi}\cdot[\vec{\Lambda}_k]}$ where $[\vec{\Lambda}_k]$ is a weight in the the E_{n+1} representation with highest weight $\vec{\Lambda}_k$.

Using the same arguments, a generic higher derivative term in d dimensions can be written as a polynomial in the Riemann curvature, the non-linearly realised field strengths and Cartan forms $\mathcal{S}_{E_{n+1}}$, but it is also multiplied by a function of the scalar fields. Assuming that the higher derivative term as a whole is invariant under an E_{n+1} transformation implies that this non-holomorphic function must be an E_{n+1} automorphic form. The automorphic form is built out of a particular representation, of E_{n+1} with highest weight $\vec{\Lambda}$ say. We write the states of this representation in the form $|\psi\rangle = n_i|\vec{\mu}_i >$ where $|\vec{\mu}_i >$ are a basis of the representation, $\vec{\mu}_i$ are the weights in the representation and n_i are integers. To be more precise it is constructed out of the non-linear representation of E_{n+1} constructed from this representation using the scalars, that is, it is constructed out of the function $|\varphi\rangle$, defined by

$$|\varphi\rangle = L(g^{-1}_{E_{n+1}})|\psi\rangle. \tag{7.15}$$

It is obvious that $|\varphi\rangle$ contains terms where the scalar fields occur in the form $e^{\vec{\phi}\cdot[\vec{\Lambda}]}$. The automorphic form is a function of $|\varphi\rangle$ and in the simplest cases it is of the generic form

$$\sum_{n_i} < \varphi|\varphi >^{-s}, \tag{7.16}$$

for some constant s. In more complicated cases the sum over the lattice in equation (7.16) satisfies some constraint, however the commonly studied automorphic forms constructed in this way always contain a term with scalar field dependence given by $e^{-\sqrt{2}s\vec{\Lambda}.\vec{\phi}}$, where $\vec{\Lambda}$ is the highest weight of the representation used to build the automorphic form. This construction is described in more detail in reference [40] and reviewed in chapter eight. The use of integers corresponds to the fact that the symmetry group for the higher derivative terms is discretised since the charges of the theory obey a quantisation condition.

We will refer to the formulation of a higher derivative term in d dimensions just described as the E_{n+1} formulation. A term in the higher derivative effective action will contain an exponential of the scalar fields $\vec{\phi}$ of the form $e^{\sqrt{2}\vec{w}\cdot\vec{\phi}}$ where $\vec{\phi}$ is the field we introduced earlier in this section. Our task is to compare this with the equivalent factor that arises in the dimensional reduction. However, in order to compare the E_{n+1} formulation of the type IIA theory in d dimensions with the dimensionally reduced formulation discussed in the previous section we need to know the relationship between the fields that occur in the dimensional reduction, namely the fields $\underline{\phi}$, ρ and ϕ, where $\underline{\phi}$ is an $n-1$-dimensional vector and those that occur in the E_{n+1} formulation, namely the $n+1$-dimensional $\vec{\phi}$. This will be given in the next section.

7.3 The E_{11} Formulation

The eleven dimensional, IIA and IIB supergravity theories, as well as the maximal type II supergravity theories in lower dimensions, can be formulated as non-linear realisations [66, 67]. The non-linear realisations of the Kac-Moody algebra E_{11}, at low levels, leads to all of these theories. As such E_{11} encodes the fields of each of these theories and provides us with a way of relating the fields in the different theories to each other. In fact the fields of these theories are in one to one correspondence with the generators of the Borel subalgebra of E_{11} in the group decomposition, explained below, appropriate to each theory.

A Kac-Moody algebra is formulated in terms of its Chevalley generators, which include those in the Cartan subalgebra denoted by $H_{\hat{a}},\ \hat{a} = 1, 2, \ldots 11$. As such, the E_{11} group element that occurs in the non-linear realisation is of the form $g_{E_{11}} = e^{\sum_{\hat{a}} \phi_{\hat{a}} H_{\hat{a}}}$ provided we restrict our attention to the part that is in the Cartan subalgebra.

The E_{11} Kac Moody algebra is encoded in the Dynkin diagram given in figure 13. The eleven

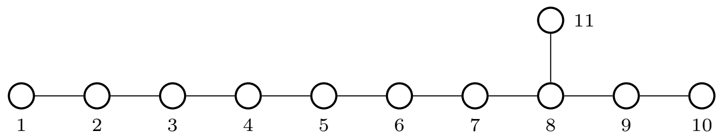

Figure 13: The E_{11} Dynkin diagram with eleven dimensional supergravity labeling

dimensional theory emerges if we decompose the E_{11} algebra in terms of the algebra that results from deleting the exceptional node labelled eleven, namely the algebra $GL(11)$. This subalgebra has the generators $K^{\hat{a}}{}_{\hat{b}},\ \hat{a}, \hat{b} = 1, \ldots, 11$, and it includes all the Cartan subalgebra generators of E_{11}; the relation being [67]

$$\begin{aligned} H_{\hat{a}} &= K^{\hat{a}}{}_{\hat{a}} - K^{\hat{a}+1}{}_{\hat{a}+1},\ \hat{a} = 1, \ldots, 10, \\ H_{11} &= -\frac{1}{3}\left(K^1{}_1 + ... + K^8{}_8\right) + \frac{2}{3}\left(K^9{}_9 + K^{10}{}_{10} + K^{11}{}_{11}\right). \end{aligned} \tag{7.17}$$

The first ten generators being the Cartan subalgebra generators of $SL(11)$.

The contribution of the $GL(10)$ subgroup to the E_{11} group element in the non-linear realisation is of the form

$$e^{x^{\hat{a}} P_{\hat{a}}} e^{h_{\hat{a}}{}^{\hat{b}} K^{\hat{a}}{}_{\hat{b}}}, \tag{7.18}$$

where we have added the space-time translations generators $P_{\hat{a}}$. This is known to give rise to eleven dimensional gravity and as a result the line in the above Dynkin diagram that is from nodes one to ten inclusive is known as the gravity line. Indeed the Cartan form for this subgroup is given by

$$g^{-1} dg = dx^{\hat{\mu}} \hat{e}_{\hat{\mu}}{}^{\hat{a}} P_{\hat{a}} + (e^{-1} de)_{\hat{a}}{}^{\hat{b}} K^{\hat{a}}{}_{\hat{b}}. \tag{7.19}$$

It turns out that $e_\mu{}^a = (e^h)_a{}^b$ is the eleven-dimensional vielbein.

Restricting to the Cartan subalgebra we may set the different formulations of the E_{11} group element to be equal to find that

$$e^{\sum_{\hat{a}} \hat{\phi}_{\hat{a}} \hat{H}_{\hat{a}}} = e^{\sum_{\hat{a}} h_{\hat{a}}{}^{\hat{a}} K^{\hat{a}}{}_{\hat{a}}}. \tag{7.20}$$

Comparing coefficients of $K^{\hat{a}}{}_{\hat{a}}$ using equation (7.17) we find the relations

$$\begin{aligned}
\phi_i &= h^1{}_1 + h^2{}_2 ... + h^i{}_i - \frac{i}{2}\sum_{j=1}^{11} h^j{}_j, \quad for \quad 1 \le i \le 8, \\
\phi_9 &= h^1{}_1 + h^2{}_2 ... + h^9{}_9 - 3\sum_{j=1}^{11} h^j{}_j, \\
\phi_{10} &= h^1{}_1 + h^2{}_2 ... + h^{10}{}_{10} - 2\sum_{j=1}^{11} h^j{}_j, \\
\phi_{11} &= -\frac{3}{2}\sum_{j=1}^{11} h^j{}_j.
\end{aligned} \tag{7.21}$$

The full non-linear realisation of E_{11} leads, at low levels and with the decomposition to $GL(11)$, to the eleven dimensional supergravity theory. However, in this case we are interested in only the fields associated with the Cartan subalgebra parts of the algebra, hence the above restriction.

Let us now consider the ten dimensional IIA theory which is obtained from eleven dimensions by dimensional reduction on a circle. In this process, the diagonal components of the eleven dimensional metric result in the diagonal components of the ten dimensional metric and a scalar ϕ, which is the dilaton of the IIA theory.

In terms of the E_{11} formulation we obtain the IIA theory by deleting nodes ten and eleven of the Dynkin diagram below (see figure 14) leaving us with the algebra $GL(10) \otimes GL(1)$ algebra; the $GL(10)$ algebra leads to ten dimensional gravity, for the same reasons as occurred above in eleven dimensions, and the $GL(1)$ factor leads to the IIA dilaton. The gravity line is now the horizontal

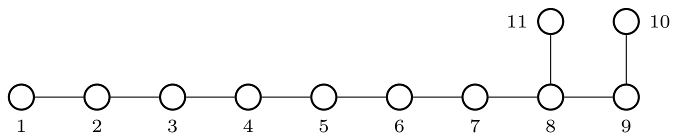

Figure 14: The E_{11} Dynkin diagram appropriate to type IIA supergravity

line of the Dynkin diagram of figure 14. The IIA supergravity theory emerges from the non-linear realisation of E_{11} with this decomposition.

Let us denote the generators of $GL(10)$ by $K^a{}_b$, $a, b = 1, \ldots, 10$ and let R be the $GL(1)$ generator. These contain the generators of the Cartan subalgebra of E_{11}. The group element in the Cartan subalgebra of E_{11} can therefore be written in the form

$$g = e^{\sum_a \tilde{h}^a{}_a K^a{}_a} e^{\sigma R}, \tag{7.22}$$

The tilde distinguishes the field from that in eleven dimensions. However, in terms of the Chevalley generators in the Cartan subalgebra of E_{11}, the group element has the same form as in eleven dimensions, namely $g = e^{\sum_{\hat{a}} \phi_{\hat{a}} H_{\hat{a}}}$.

It turns out that the Cartan sub-algebra generators $H_{\hat{a}}$ of the E_{11} algebra and those in the $GL(10) \otimes GL(1)$ algebra are related by [66]

$$\begin{aligned} H_a &= K^a{}_a - K^{a+1}{}_{a+1}, \quad a = 1, ..., 9, \\ H_{10} &= -\frac{1}{8}\left(K^1{}_1 + ... + K^9{}_9\right) + \frac{7}{8} K^{10}{}_{10} - \frac{3}{2} R, \\ H_{11} &= -\frac{1}{4}\left(K^1{}_1 + ... + K^8{}_8\right) + \frac{3}{4}\left(K^9{}_9 + K^{10}{}_{10}\right) + R. \end{aligned} \tag{7.23}$$

Equating the group element g in the Cartan subalgebra written in terms of the two different set of generators we find that

$$\begin{aligned} g = e^{\sum_{a=1}^{10} \tilde{h}^a{}_a K^a{}_a} e^{\sigma R} &= e^{\phi_1\left(K^1{}_1 - K^2{}_2\right)} ... e^{\phi_9\left(K^9{}_9 - K^{10}{}_{10}\right)} \\ \times\, e^{\phi_{10}\left(-\frac{1}{8}\left(K^1{}_1 + ... + K^9{}_9\right) + \frac{7}{8} K^{10}{}_{10} - \frac{3}{2} R\right)} & e^{\phi_{11}\left(-\frac{1}{4}\left(K^1{}_1 + ... + K^8{}_8\right) + \frac{3}{4}\left(K^9{}_9 + K^{10}{}_{10}\right) + R\right)}. \end{aligned} \tag{7.24}$$

Comparing the coefficients of the generators R and $K^a{}_a$ we find that

$$\begin{aligned} \sigma &= -\frac{3}{2}\phi_{10} + \phi_{11}, \\ \tilde{h}^1{}_1 &= \phi_1 - \frac{1}{8}\phi_{10} - \frac{1}{4}\phi_{11}, \\ \tilde{h}^i{}_i &= -\phi_{i-1} + \phi_i - \frac{1}{8}\phi_{10} - \frac{1}{4}\phi_{11}, \quad for \ \ 2 \le i < 9, \\ \tilde{h}^9{}_9 &= -\phi_8 + \phi_9 - \frac{1}{8}\phi_{10} + \frac{3}{4}\phi_{11}, \\ \tilde{h}^{10}{}_{10} &= -\phi_9 + \phi_{10} + \frac{7}{8}\phi_{10} + \frac{3}{4}\phi_{11}. \end{aligned} \tag{7.25}$$

We now consider the d dimensional maximal supergravity theory. In the previous section we dimensionally reduced the IIA theory using the ansatz of equation (7.10) to find the IIA dilaton ϕ of the original theory, and the $10-d$ scalars $\underline{\phi}$ arising from the diagonal components of the metric G_{ij} and the field ρ.

From the E_{11} perspective, the d dimensional type II supergravity theory is found by writing the E_{11} Dynkin diagram in the form given in figure 15 below. Deleting node d we find the residual algebra $E_{n+1} \otimes GL(d)$; the latter algebra leads in the non-linear realisation to d-dimensional gravity and the former algebra is the U-duality group. Decomposing the E_{11} non-linear realisation into representations of $E_{n+1} \otimes GL(d)$ we find, at low levels, the field content of the maximal supergravity

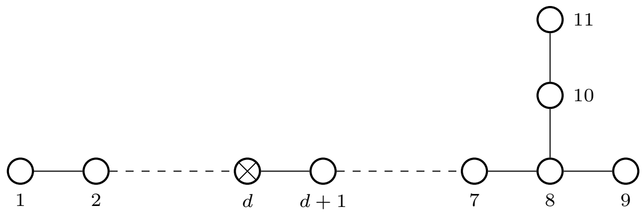

Figure 15: The E_{11} Dynkin diagram appropriate to d dimensional maximal supergravity

theory in d dimensions and indeed the supergravity theory itself. We can further delete nodes ten and eleven, corresponding to the dimensional reduction of the IIA theory, and then we find the subalgebra $GL(d) \otimes SL(n) \otimes GL(1) \otimes GL(1)$, in other words the E_{n+1} algebra has been decomposed into $SL(n) \otimes GL(1) \otimes GL(1)$. The $SL(n)$ arises from the line in the Dynkin diagram from nodes $d+1$ to 9 inclusive while the $GL(1) \otimes GL(1)$ factors are essentially the Cartan subalgebra elements associated with nodes ten and eleven. Since the $SL(n)$ algebra is part of the algebra, gravity line, of the IIA theory we conclude that the $SL(n)$ symmetry is the part preserved by dimensional reduction on the n torus as discussed in section 7.1. We note that $SL(n) \otimes GL(1) \otimes GL(1)$ contains all the Cartan subalgebra elements of E_{11} and that one of the $GL(1)$ factors is the $GL(1)$ symmetry of the IIA theory and that the two factors lead in the non-linear realisation to the fields ϕ and ρ.

The dimensional reduction ansatz of equation (7.10) is implemented in terms of E_{11} by rewriting the group element of the IIA theory of equation (7.22) in the form

$$g = e^{\sum_{a=1}^{d} \dot{h}^a{}_a K^a{}_a + e_1 \rho \sum_{a=1}^{d} K^a{}_a} e^{\sum_{i=d+1}^{10} \dot{h}^i{}_i K^i{}_i + e_2 \rho \sum_{i=d+1}^{10} K^i{}_i} e^{\sigma R}. \tag{7.26}$$

Here the $K^a{}_b$, $a, b = 1, \ldots d$, are the generators of the $GL(d)$ algebra associated with d-dimensional gravity, $K^i{}_j$, $i, j = 1, \ldots, n$ are the generators of $SL(n) \otimes GL(1)$ and e_1 and e_2 are constants. We have put a dot on the h fields to distinguish them from the analogous fields used earlier in ten and eleven dimensions. Taking into account the introduction of the field ρ we set

$$\dot{h}^{d+1}{}_{d+1} + \dot{h}^{d+2}{}_{d+2} + ... + \dot{h}^{10}{}_{10} = 0. \tag{7.27}$$

Introducing translation generators in d-dimensional space-time and internal space into the group element by including the factor $e^{x^a P_a + x^i P_i}$ and computing the Cartan forms we find the terms involving these new generators are given by the expression $g^{-1}\left(dx^a P_a + dx^i P_i\right) g$ which implies the identification

$$\sum_{a=1}^{d} \left(e^{\dot{h}^a{}_a + e_1 \rho}\right) dx^a P_a + \sum_{i=d+1}^{10} \left(e^{\dot{h}^i{}_i + e_2 \rho}\right) dx^i P_i = e^{\alpha\rho} e_\mu{}^a dx^\mu P_a + e^{\beta\rho} e_j{}^i dx^j P_i, \tag{7.28}$$

Taking $e_1 = \alpha$, $e_2 = \beta$ we do indeed recover the vielbeins as they appear in the dimensional reduction ansatz of equation (7.10) provided we identify $e_\mu{}^a = (e^{\dot{h}})_\mu{}^a$ with the vielbein in d-dimensional space-time and $e_i{}^{\underline{j}} = (e^{\dot{h}})_i{}^{\underline{j}}$ with the vielbein in the n-dimensional internal space. Equation (7.27) implies that this latter vielbein satisfies the constraint $\det e_i{}^{\underline{j}} = 1$ as required in the dimensional reduction ansatz. To find equation (7.28) we have dropped various factors involving exponentials of the trace of h as these are interpreted as $\det(e)$.

We now discuss the E_{n+1} formulation of section 7.2 from the view point of the E_{11} non-linear realisation. For simplicity we will consider only the case $d \leq 7$. We saw from figure 15 that deleting node d leads to the algebra $GL(d) \otimes E_{n+1}$. By examining the E_{11} algebra one can find the generators of $GL(d) \otimes E_{n+1}$ in terms of those of $K^a{}_b$, $a, b = 1, \dots, 11$, $R^{a_1 a_2 a_3}$ etc. One finds that the generators of $GL(d)$ are $K^a{}_b$, $a, b = 1, \dots, d$ and the Chevalley generators $T_a, a = d+1, \dots, 11$ in the Cartan subalgebra generators of E_{n+1} are given by

$$\begin{aligned}
T_{d+1} &= \dot{K}^{d+1}{}_{d+1} - \dot{K}^{d+2}{}_{d+2}, \\
&\vdots \\
T_8 &= \dot{K}^8{}_8 - \dot{K}^9{}_9, \\
T_9 &= \dot{K}^9{}_9 - \dot{K}^{10}{}_{10}, \\
T_{10} &= -\frac{1}{8}\left(\dot{K}^{d+1}{}_{d+1} + ... + \dot{K}^9{}_9\right) + \frac{7}{8}\dot{K}^{10}{}_{10} - \frac{3}{2}\dot{R}, \\
T_{11} &= -\frac{1}{4}\left(\dot{K}^{d+1}{}_{d+1} + ... + \dot{K}^8{}_8\right) + \frac{3}{4}\left(\dot{K}^9{}_9 + \dot{K}^{10}{}_{10}\right) + \dot{R}.
\end{aligned} \tag{7.29}$$

where $\dot{K}^a{}_b = K^a{}_b - \frac{1}{d-2}\delta^a_b \sum_{c=1}^d K^c{}_c$ for $a, b = d+1, \dots, 10$ and $\dot{R} = R$. We note that the generators $\dot{K}^a{}_b$ obey the necessary condition $[\dot{K}^a{}_b, P_c] = 0$ for $a, b = d+1, ..., 10$ and $c = 1, \dots, d$. In this last equation we have used the commutator $[K^a{}_b, P_c] = -\delta^a_c P_b + \frac{1}{2}\delta^a_b P_c$. It is straightforward to verify that $T_a = H_a$, $a = d+1, \dots, 11$, where these H_a are the Chevalley generators of E_{11} given in equation (7.17), or equivalently from the IIA viewpoint in equation (7.23). The E_{11} group element written in a way that displays the $GL(d) \otimes E_{n+1}$ decomposition required in d dimensions and restricted to lie in the Cartan subalgebra can be written as

$$g = e^{\sum_{a=1}^d \dot{h}^a{}_a K^a{}_a} e^{\sum_{a=d+1}^{11} \varphi_a T_a}. \tag{7.30}$$

We have used that the $GL(d)$ generators are $K^a{}_b$ for $a, b = 1, \dots, d$ and denoted the E_{n+1} fields by φ_a for $a = d+1, \dots, 11$.

We can now equate the two different ways of expressing the E_{11} group element given in equations (7.26) and (7.29), that is the one that implements the dimensional reduction from the IIA theory to the one that has the $GL(d) \otimes E_{n+1}$ decomposition in d dimensions. Using equations (7.29) and

(7.23) and keeping only terms involving $K^a{}_a, a = d+1, \ldots, 11$ we find the equation

$$
\begin{aligned}
&e^{\left(\dot{h}^{d+1}{}_{d+1}+e_2\rho\right)K^{d+1}{}_{d+1}}\ldots e^{\left(\dot{h}^{9}{}_{9}+e_2\rho\right)K^{9}{}_{9}}e^{\left(\dot{h}^{10}{}_{10}+e_2\rho\right)K^{10}{}_{10}}e^{\sigma R}\\
&= e^{\varphi_{d+1}\left(K^{d+1}{}_{d+1}-K^{d+2}{}_{d+2}\right)}\ldots e^{\varphi_8\left(K^{8}{}_{8}-K^{9}{}_{9}\right)}e^{\varphi_9\left(K^{9}{}_{9}-K^{10}{}_{10}\right)}\\
&\times e^{\varphi_{10}\left(-\frac{1}{8}\left(K^{d+1}{}_{d+1}+\ldots+K^{9}{}_{9}\right)+\frac{7}{8}K^{10}{}_{10}-\frac{3}{2}R\right)}e^{\varphi_{11}\left(-\frac{1}{4}\left(K^{d+1}{}_{d+1}+\ldots+K^{8}{}_{8}\right)+\frac{3}{4}\left(K^{9}{}_{9}+K^{10}{}_{10}\right)+R\right)}.
\end{aligned}
\tag{7.31}
$$

Equating the coefficients of the generators we find the relations

$$
\begin{aligned}
\dot{h}^{d+1}{}_{d+1}+e_2\rho &= \varphi_{d+1}-\frac{1}{8}\varphi_{10}-\frac{1}{4}\varphi_{11},\\
\dot{h}^{d+2}{}_{d+2}+e_2\rho &= -\varphi_{d+1}+\varphi_{d+2}-\frac{1}{8}\varphi_{10}-\frac{1}{4}\varphi_{11},\\
&\ldots\\
\dot{h}^{8}{}_{8}+e_2\rho &= -\varphi_7+\varphi_8-\frac{1}{8}\varphi_{10}-\frac{1}{4}\varphi_{11},\\
\dot{h}^{9}{}_{9}+e_2\rho &= -\varphi_8+\varphi_9-\frac{1}{8}\varphi_{10}+\frac{3}{4}\varphi_{11},\\
\dot{h}^{10}{}_{10}+e_2\rho &= -\varphi_9+\frac{7}{8}\varphi_{10}+\frac{3}{4}\varphi_{11},\\
\sigma &= -\frac{3}{2}\varphi_{10}+\varphi_{11}.
\end{aligned}
\tag{7.32}
$$

Solving these equations for the E_{11} fields we find that

$$
\begin{aligned}
\varphi_i &= \dot{h}^{d+1}{}_{d+1}+\dot{h}^{d+2}{}_{d+2}+\ldots+\dot{h}^{i}{}_{i}+(n-10+i)\frac{8}{8-n}e_2\rho, \quad d+1\le i<8,\\
\varphi_9 &= \dot{h}^{d+1}{}_{d+1}+\dot{h}^{d+2}{}_{d+2}+\ldots+\dot{h}^{9}{}_{9}+\frac{5n-8}{8-n}e_2\rho-\frac{1}{4}\sigma,\\
\varphi_{10} &= -\frac{1}{2}\sigma+\frac{2}{8-n}ne_2\rho,\\
\varphi_{11} &= \frac{1}{4}\sigma+\frac{3}{8-n}ne_2\rho.
\end{aligned}
\tag{7.33}
$$

In this section we have formulated the E_{11} algebra in terms of the Chevalley generators, in particular the Cartan subalgebra generators H_a , $a = 1, \ldots, 11$, however, in section 7.2 we used the Cartan-Weyl basis with generators $H_i, i = 1, \ldots, 11$. The advantage of the latter basis is that acting on a state $|\vec{\Lambda}>$ of weight $\vec{\Lambda}_i$, the generators H_i, almost by definition, read off the weight i.e. $H_i|\Lambda >= \Lambda_i|\vec{\Lambda}>$. The two sets of generators are related by $\alpha_a^i H_i = H_a$ where $\vec{\alpha}_a$ are the simple roots and α_a^i is the i'th component. If we denote the fields in the Cartan-Weyl basis by $\tilde{\varphi}_a$. The corresponding fields are related by $\tilde{\varphi}^i H_i = \varphi^a H_a$ which implies the relation

$$\tilde{\varphi}^i = \varphi^a \alpha_a^i. \tag{7.34}$$

In addition, the fields in the E_{11} group element $\tilde{\varphi}_i$ are equal to those in the automorphic form φ_i, up to a numerical factor. We see, through comparing the normalisations of the fields in the the

E_{11} group element $e^{\sum_i \tilde{\varphi}_i H_i}$ and the automorphic form group element $e^{-\frac{1}{\sqrt{2}}\vec{\varphi}.\vec{H}}$, that

$$\vec{\varphi} = \left(-\sqrt{2}\tilde{\varphi}_1, -\sqrt{2}\tilde{\varphi}_2, -\sqrt{2}\underline{\tilde{\varphi}}\right). \tag{7.35}$$

Using equation (D.15) of appendix and equation (7.34) we then find that the components of the E_{11} group element fields $\tilde{\varphi}_i$ in Cartan-Weyl Basis are

$$\begin{aligned}
\tilde{\varphi}^1 &= x\varphi^{10} = x\left(-\frac{1}{2}\sigma + \left(\frac{2}{8-n}\right)ne_2\rho\right), \\
\tilde{\varphi}^2 &= -\frac{\underline{\lambda}_{n-2}.\underline{\lambda}_{n-1}}{y}\varphi_{10} + y\varphi_{11} \\
&= -\frac{\underline{\lambda}_{n-2}.\underline{\lambda}_{n-1}}{y}\left(-\frac{1}{2}\sigma + \left(\frac{2}{8-n}\right)ne_2\rho\right) + y\left(\frac{1}{4}\sigma + \left(\frac{3}{8-n}\right)ne_2\rho\right),
\end{aligned} \tag{7.36}$$

and

$$\underline{\tilde{\varphi}} = \sum_{i=d+1}^{9} \varphi_i \underline{\alpha}_{i-d} - \varphi_{10}\underline{\lambda}_{n-1} - \varphi_{11}\underline{\lambda}_{n-2}. \tag{7.37}$$

Note that

$$\underline{\tilde{\varphi}}.\underline{\alpha}_i = \dot{h}^i{}_i - \dot{h}^{i+1}{}_{i+1}, \quad \underline{\tilde{\varphi}}.\underline{\lambda}_j = \sum_{i=d+1}^{d+j} \dot{h}^i{}_i. \tag{7.38}$$

7.4 Constraints on the Automorphic Forms

In this section we will compare the E_{n+1} formulation in d dimensions given in section 7.2 with the results of section 7.1 found by dimensionally reducing the IIA theory in ten dimensions and find constraints on the automorphic forms. In order to carry out the comparison we will use the field relations of the last section. The field strengths occurred in the E_{n+1} formulation as the non-linear representations $\mathcal{F}_{E_{n+1}}$ given in equation (7.14) while the derivatives of the scalars occur in the Cartan forms $\mathcal{S}_{E_{n+1}}$. These are constructed using the group element $g_{E_{n+1}}$, however, this is just the E_{11} group element restricted to lie in the subalgebra E_{n+1} and it is given below equation (7.28). We noted that if one deletes nodes 10 and 11 in the E_{11} Dynkin diagram the E_{n+1} algebra is reduced to $SL(n) \times GL(1) \times GL(1)$.

In the dimensional reduction of the IIA theory we found a manifest $SL(n) \otimes GL(1)$ symmetry; the first factor arises from the diffeomorphisms preserved by the torus while the second factor is the $GL(1)$ symmetry of the IIA theory in ten dimensions. As such, the field strengths that appear in the dimensional reduction could be expressed in terms of the non-linear representation of $SL(n) \otimes GL(1)$ denoted by $\mathcal{F}_{SL(n)\otimes GL(1)}$ and the derivatives of the scalars in terms of the Cartan forms $S_{SL(n)\otimes GL(1)}$.

Deleting nodes ten and eleven of the Dynkin diagram of figure 15 we find that E_{n+1} decomposes into $SL(n) \otimes GL(1) \otimes GL(1)$ and one can carry out the decomposition of the non-linear representations that occur in the E_{n+1} formulation. Clearly, the non-linear representation of the field strengths $\mathcal{F}_{E_{n+1}}$ will decompose into the non-linear representations $\mathcal{F}_{SL(n)\otimes GL(1)}$ with appropriate factors corresponding to the additional $GL(1)$. The same discussion applies to the derivative of the scalars which appear in $\mathcal{S}_{E_{n+1}}$ and $\mathcal{S}_{SL(n)\otimes GL(1)}$. Given a particular term in the higher derivative effective action the $SL(n)\otimes GL(1)$ parts will automatically agree and it is with the comparison of the $GL(1)$ factors that we find non-trivial results.

It would be instructive to systematically carry out the decomposition of E_{n+1} formulation when decomposed to $SL(n)\times GL(1)\times GL(1)$, but for our present purposes it suffices to carry it out for the generators that belong to the Cartan subalgebra. With this restriction the group element of E_{n+1} is given, below equation (7.28), by $g_{E_{n+1}} = e^{\sum_a H_a \dot{\varphi}_a}$, but an equivalent formulation, in terms of the field variables associated with dimensional reduction, is given in equation (7.31). Matching these we found in equations (7.32) and (7.33) how the fields $\dot{\varphi}_a,\ a = d+1,\ldots,11$ correspond to the fields $h^{d+1}{}_{d+1}, ..., h^{10}{}_{10}$, ρ and ϕ found in the dimensional reduction of the IIA theory. The additional $GL(1)\otimes GL(1)$ group found in the reduction then corresponds to the Cartan subalgebra generators H_{10} and H_{11} or from the dimensional reduction viewpoint to the fields ρ and ϕ.

We now consider the decomposition in more detail. One may write any root of E_{n+1} in terms of its simple roots:

$$\vec{\alpha} = m_c\vec{\alpha}_{n+1} + n_c\vec{\alpha}_n + \sum_{i=1}^{n-1} m_i\vec{\alpha}_i = n_c\left(x, -\frac{\underline{\lambda_{n-2}}\cdot\underline{\lambda_{n-1}}}{y}, \underline{0}\right) + m_c\left(0, y, \underline{0}\right) - \vec{\lambda} \tag{7.39}$$

where $\vec{\lambda} = m_c\underline{\lambda}_{n-2} + n_c\underline{\lambda}_{n-1} - \sum_{i=1}^{n-1} k_i\underline{\vec{\alpha}_i}$. The roots of E_{n+1} are labelled by the integers m_c, n_c which are referred to as the levels. If a representation of $SL(n)$ occurs in the decomposition of the adjoint representation of E_{n+1} then its highest weight must appear on the right-hand side as one of the $\underline{\lambda}$'s. We can examine which representations occur level by level. At level $n_c = m_c = 0$ one obviously finds the adjoint representation of $SL(n)$. At higher levels the highest weights, and so representations, of $SL(n)$ that occur are given in the table below

$$\begin{array}{cccc} m_c=1,\ n_c=0 & m_c=0,\ n_c=1 & m_c=1,\ n_c=1 & m_c=2,\ n_c=1 \\ \underline{\lambda}_2 & \underline{\lambda}_1 & \underline{\lambda}_3 & \underline{0} \\ & & & \\ m_c=3,\ n_c=1 & m_c=2,\ n_c=2 & m_c=3,\ n_c=2 & \\ \underline{0} & \underline{\lambda}_6 & \underline{\lambda}_1. & \end{array} \tag{7.40}$$

As such one finds that the weights in the adjoint representation of E_{n+1} are given by

$$\begin{aligned}
&\left(0,0,\left[\underline{\alpha}_1+\ldots+\underline{\alpha}_{n-1}\right]\right),\left(0,y,[\underline{\lambda}_2]\right),\left(x,-\frac{\underline{\underline{\lambda_{n-2}\cdot\lambda_{n-1}}}}{y},[\underline{\lambda}_1]\right),\\
&\left(x,-\frac{\underline{\underline{\lambda_{n-2}\cdot\lambda_{n-1}}}}{y}+y,[\underline{\lambda}_3]\right),\left(x,-\frac{\underline{\underline{\lambda_{n-2}\cdot\lambda_{n-1}}}}{y}+2y,\underline{0}\right),\\
&\left(x,-\frac{\underline{\underline{\lambda_{n-2}\cdot\lambda_{n-1}}}}{y}+3y,\underline{0}\right),\left(2x,-2\frac{\underline{\underline{\lambda_{n-2}\cdot\lambda_{n-1}}}}{y}+2y,[\underline{\lambda}_6]\right),\\
&\left(2x,-2\frac{\underline{\underline{\lambda_{n-2}\cdot\lambda_{n-1}}}}{y}+3y,[\underline{\lambda}_1]\right).
\end{aligned} \tag{7.41}$$

The Cartan form $\mathcal{S}_{E_{n+1}}$ belongs to the adjoint representation of E_{n+1} and at level $m_c = n_c = 0$ decompose into the Cartan forms of $SL(n)$. Using the decomposition of equation (7.41) we see that at higher levels they decompose as follows

$$\begin{array}{cccc}
m_c=1,\ n_c=0 & m_c=0,\ n_c=1 & m_c=1,\ n_c=1 & m_c=2,\ n_c=1\\
\mathcal{S}_{SL(n)i_1i_2} & \mathcal{S}_{SL(n)i} & \mathcal{S}_{SL(n)i_1i_2i_3} & \mathcal{S}_{SL(n)i_1i_2\ldots i_n}\\
\\
m_c=3,\ n_c=1 & m_c=2,\ n_c=2 & m_c=3,\ n_c=2 & \\
\mathcal{S}_{SL(n)i_1i_2\ldots i_n} & \mathcal{S}_{SL(n)i_1\ldots i_6} & \mathcal{S}_{SL(n)i.} &
\end{array} \tag{7.42}$$

We noted previously that the Cartan form $\mathcal{S}_{E_{n+1}}$ contains a dependence on the scalars $\vec{\phi}$ in the form factor $e^{\frac{1}{\sqrt{2}}\vec{\phi}.\vec{\alpha}}$. Under the decomposition we find the $SL(n)\otimes GL(1)$ Cartan forms $S_{SL(n)\otimes GL(1)}$ and exponentials in ρ. Using equations (7.36)-(7.38), $(D.15)$ and $(D.18)$ we find that the latter factors at level m_c, n_c are

$$e^{(2m_c+n_c)\alpha\rho\left(\frac{8-n}{n}\right)}. \tag{7.43}$$

We now consider the terms that result from the dimensional reduction from the IIA theory using the discussion of section 7.1. The ten dimensional origins of the decomposition of the adjoint representation of E_{n+1} at each level may be found by examining the $SL(n)$ and space-time index structure. In particular,we see that the $SL(n)\otimes GL(1)$ Cartan forms $\mathcal{S}_{SL(n)\otimes GL(1)}$ at levels $(m_c=0,\ n_c=1)$, $(m_c=1,\ n_c=0)$ and $(m_c=1,\ n_c=1)$ come from the dimensional reduction of the two-form field strength $\tilde{\mathcal{F}}_{a\underline{i}_1\underline{i}_2}$, three form field strength $\tilde{\mathcal{F}}_{a\underline{i}_1\underline{i}_2\underline{i}_3}$ and four-form field strength $\tilde{\mathcal{F}}_{a\underline{i}_1\underline{i}_2\underline{i}_3\underline{i}_4}$ respectively. The Cartan forms, at higher levels, are associated with the dimensional reduction of the dualised two, three and four-form fields strengths for levels $(m_c=3,\ n_c=1)$, $(m_c=2,\ n_c=2)$ and $(m_c=2,\ n_c=1)$ respectively, along with the dualised graviphoton at level $(m_c=3,\ n_c=2)$. We note that the dualised four-form only appears as a Cartan form of $SL(n)\otimes GL(1)$ in $d=5$, while the dualised three form is only present as a Cartan form of $SL(n)\otimes GL(1)$ in $d=4$. While the dualised graviphoton is a Cartan form of $SL(n)\otimes GL(1)$ only in $d=3$ and we also find the dualised two-form is also realised as a Cartan form of $SL(n)$.

The Cartan forms of $SL(n)\otimes GL(1)$, arising upon dimensional reduction, carry one d dimensional space-time index and $(2m_c+n_c)$ internal indices. Therefore, each Cartan form of $SL(n)\otimes GL(1)$, at a given level, occurs with an exponential of ρ which is given by

$$e^{-\rho(\alpha+(2m_c+n_c)\beta)} = e^{(2m_c+n_c)\alpha\rho\left(\frac{8-n}{n}\right)}e^{-\alpha\rho}. \tag{7.44}$$

Comparing with the result, given in equation (7.43), of the E_{n+1} formulation we find a surplus factor of $e^{-\alpha\rho}$ multiplying the dimensionally reduced term. We note that the factors involving ϕ and $\underline{\phi}$ will match automatically.

To treat the other building blocks in the same way we must learn how to decompose more general representations of E_{n+1} into those of $SL(n)\times GL(1)\times GL(1)$. To do this we use the technique of reference [74]. If one wants to consider the representation of E_{n+1} with highest weight Λ_i, associated with the node labeled i, we add a new node, denoted $\star$, to the E_{n+1} Dynkin diagram which is connected to the node labeled i by a single line to construct the Dynkin diagram for an enlarged algebra of rank $n+2$. Deleting the $\star$-node we recover the E_{n+1} Dynkin diagram and the representation of E_{n+1} with highest weight Λ_i is found in the adjoint representation of the enlarged algebra provided we keep only contributions at level $n_\star = 1$. Thus we find the decomposition of the representation of E_{n+1} with highest weight Λ_i into representations of $SL(n)\times GL(1)\times GL(1)$ by decomposing the adjoint representation of the enlarged algebra but deleting the additional node and keeping only contributions with $n_\star = 1$ and deleting nodes 10 and 11 but keeping all levels of m_c and n_c.

In the E_{n+1} formulation of the effective action in d dimensions, the one-form gauge field, out of which the two-form field strengths are constructed, appear in the representation with highest weight $\vec{\Lambda}_1$. The $\vec{\Lambda}_1$ representation of E_{n+1} may be decomposed into representations of $SL(n)$, with an associated type IIA dilaton weight, level by level. At level (m_c, n_c) one finds

$$\begin{array}{cccc} m_c=0,\ n_c=0 & m_c=1,\ n_c=0 & m_c=0,\ n_c=1 & m_c=1,\ n_c=1 \\ \underline{\lambda}_1 & \underline{\lambda}_{n-1} & \underline{0} & \underline{\lambda}_{n-2} \\ & & & \\ m_c=2,\ n_c=1 & m_c=2,\ n_c=2 & m_c=3,\ n_c=1 & m_c=3,\ n_c=2 \\ \underline{\lambda}_{n-4} & \underline{\lambda}_{n-5} & \underline{\lambda}_{n-6} & \underline{\lambda}_{n-1}. \end{array} \tag{7.45}$$

Therefore, the weights of the $\vec{\Lambda}_1$ representation are

$$\left(\frac{1}{2x}, \frac{\underline{\lambda}_1\cdot\underline{\lambda}_{n-2}}{y}, [\underline{\lambda}_1]\right), \left(\frac{1}{2x}, \frac{\underline{\lambda}_1\cdot\underline{\lambda}_{n-2}}{y} - y, [\underline{\lambda}_{n-1}]\right), \left(\frac{1}{2x} - x, \frac{\underline{\lambda}_1\cdot\underline{\lambda}_{n-2}}{y} + \frac{\underline{\lambda}_{n-1}\cdot\underline{\lambda}_{n-2}}{y}, [\underline{0}]\right), \tag{7.46}$$

$$\left(\frac{1}{2x} - x, \frac{\underline{\lambda}_1\cdot\underline{\lambda}_{n-2}}{y} - y + \frac{\underline{\lambda}_{n-1}\cdot\underline{\lambda}_{n-2}}{y}, [\underline{\lambda}_{n-2}]\right), \tag{7.47}$$

$$\left(\frac{1}{2x}-x,\frac{\underline{\lambda}_1\cdot\underline{\lambda}_{n-2}}{y}-2y+\frac{\underline{\lambda}_{n-1}\cdot\underline{\lambda}_{n-2}}{y},\left[\underline{\lambda}_{n-4}\right]\right), \tag{7.48}$$

$$\left(\frac{1}{2x}-2x,\frac{\underline{\lambda}_1\cdot\underline{\lambda}_{n-2}}{y}-2y+2\frac{\underline{\lambda}_{n-1}\cdot\underline{\lambda}_{n-2}}{y},\left[\underline{\lambda}_{n-5}\right]\right), \tag{7.49}$$

$$\left(\frac{1}{2x}-x,\frac{\underline{\lambda}_1\cdot\underline{\lambda}_{n-2}}{y}-3y+\frac{\underline{\lambda}_{n-1}\cdot\underline{\lambda}_{n-2}}{y},\left[\underline{\lambda}_{n-6}\right]\right), \tag{7.50}$$

$$\left(\frac{1}{2x}-2x,\frac{\underline{\lambda}_1\cdot\underline{\lambda}_{n-2}}{y}-3y+2\frac{\underline{\lambda}_{n-1}\cdot\underline{\lambda}_{n-2}}{y},\left[\underline{\lambda}_{n-1}\right]\right). \tag{7.51}$$

From the weights, we see that the corresponding two-form field strengths, at each level, are

$$\begin{array}{cccc}
m_c=0,\ n_c=0 & m_c=1,\ n_c=0 & m_c=0,\ n_c=1 & m_c=1,\ n_c=1 \\
\mathcal{F}^{i}_{a_1a_2} & \mathcal{F}_{a_1a_2i} & \mathcal{F}_{a_1a_2} & \mathcal{F}_{a_1a_2i_1i_2} \\
\\
m_c=2,\ n_c=1 & m_c=2,\ n_c=2 & m_c=3,\ n_c=1 & m_c=3,\ n_c=2 \\
\mathcal{F}_{a_1a_2i_1\ldots i_4} & \mathcal{F}_{a_1a_2i_1\ldots i_5} & \mathcal{F}_{a_1a_2i_1\ldots i_n} & \mathcal{F}_{a_1a_2i}.
\end{array} \tag{7.52}$$

After dualisation, any two-form field strength will appear as a one-form field strength in $d=3$ dimensions therefore we only need to consider two-form field strengths in $d\geq 4$ dimensions. One finds the maximum level that contributes is $(m_c=3,\ n_c=2)$, in the remaining dimensions any level (m_c,n_c) listed in the above decomposition will appear in d dimensions if $(2m_c+n_c-1)\leq n$. The two-form field strength at level $(m_c,\ n_c)$ arises through the dimensional reduction of the metric at level $(0,0)$, three-form field strength at level $(1,0)$, two-form field strength at level $(0,1)$ and four-form field strength at level $(1,1)$. The higher levels in the decomposition of the representation with highest weight $\vec{\Lambda}_1$ are associated with the dimensional reduction of the dualised field strengths and the graviphoton.

A two-form field strength in some representation of $SL(n)$ at level $(m_c,\ n_c)$ in the E_{n+1} formulation of the IIA theory appears multiplied by the factor

$$e^{-\frac{8}{n}\alpha\rho-(2m_c+n_c)\left(\frac{8-n}{n}\right)\alpha\rho}, \tag{7.53}$$

where the factors associated with $SL(n)$ fields $\underline{\phi}$ and the IIA dilaton ϕ match those found upon dimensional reduction. Comparing the volume with the dimensionally reduced two-form field strengths, which carry two d dimensional indices and $2m_c+n_c-1$ internal indices and as a result appear multiplied by the factor

$$e^{-\rho(2\alpha+(2m_c+n_c-1)\beta)}=e^{-\alpha\rho}e^{-\frac{8}{n}-(2m_c+n_c)\left(\frac{8-n}{n}\right)\alpha\rho}, \tag{7.54}$$

we find that the two-form field strengths in the dimensionally reduced type IIA effective action carry an additional factor of $e^{-\alpha\rho}$.

Three-form field strengths appear in the type IIA effective action in $d \geq 6$ dimensions. In the E_{n+1} formulation, the two-form gauge fields, from which the three-form field strengths are constructed, lie in the representation with highest weight $\vec{\Lambda}_n$. The $\vec{\Lambda}_n$ representation decomposes into representations of $SL(n)$ with an associated type IIA dilaton weight, at level $(m_c,\ n_c)$, in the following way

$$\begin{array}{cccc} m_c = 0,\ n_c = 0 & m_c = 0,\ n_c = 1 & m_c = 1,\ n_c = 1 & m_c = 1,\ n_c = 2 \\ \underline{0} & \underline{\lambda}_{n-1} & \underline{\lambda}_{n-3} & \underline{0}. \end{array} \tag{7.55}$$

This decomposition leads one to observe that the weights in the $\vec{\Lambda}_{n+1}$ representation of E_{n+1} are

$$\left(\frac{1}{x}, 0,\ \underline{0}\right), \left(\frac{1}{x} - x,\ \frac{\underline{\lambda}_{n-2}\cdot\underline{\lambda}_{n-1}}{y}, \left[\underline{\lambda}_{n-1}\right]\right), \left(\frac{1}{x} - x,\ \frac{\underline{\lambda}_{n-2}\cdot\underline{\lambda}_{n-1}}{y} - y, \left[\underline{\lambda}_{n-3}\right]\right), \tag{7.56}$$

$$\left(\frac{1}{x} - 2x,\ \frac{2\underline{\lambda}_{n-2}\cdot\underline{\lambda}_{n-1}}{y} - y,\ \underline{0}\right). \tag{7.57}$$

The three-form field strengths, at level $(m_c,\ n_c)$, are

$$\begin{array}{cccc} m_c = 0,\ n_c = 0 & m_c = 0,\ n_c = 1 & m_c = 1,\ n_c = 1 & m_c = 1,\ n_c = 2 \\ \mathcal{F}_{a_1a_2a_3} & \mathcal{F}_{a_1a_2a_3i} & \mathcal{F}_{a_1a_2i_1i_2i_3} & \mathcal{F}_{a_1a_2a_3}. \end{array} \tag{7.58}$$

Any three-form may be dualised to a lower degree form in $d \leq 5$, therefore we need only consider three-form field strengths for $n \leq 4$. For $n = 4$ all of the three-form field strengths listed above are present. For $n < 4$ a three-form field strength, at level $(m_c,\ n_c)$, will be present if $(2m_c + n_c \leq n)$. The origin of the three-form field strengths is clear, the three-form field strength at level $(0,\ 0)$ is the dimensionally reduced three-form field strength, while the three-form field strength at level $(0,\ 1)$ is the dimensionally reduced four-form field strength. The remaining two levels are associated with the duals of the dimensionally reduced three and four-form field strengths. The decomposition of the representation of E_{n+1} with highest weight $\vec{\Lambda}_n$, at level $(m_c,\ n_c)$, leads to the E_{n+1} formulation of the non-linearly realised three-form field strengths containing the factor of

$$e^{\left(-2+(2m_c+n_c)\left(\frac{8-n}{n}\right)\right)\alpha\rho}, \tag{7.59}$$

again, we find the factors involving the IIA dilaton ϕ and the $SL(n)$ fields $\underline{\phi}$ agree with the dimensionally reduced formulation. However, the three-form field strengths in the dimensionally reduced formulation come with three space-time indices and $2m_c + n_c$ internal indices, therefore they carry a factor of

$$e^{-\rho(3\alpha+(2m_c+n_c)\beta)} = e^{-\alpha\rho}e^{\left(-2+(2m_c+n_c)\left(\frac{8-n}{n}\right)\right)\alpha\rho}. \tag{7.60}$$

Comparing the ρ factor of the E_{n+1} formulation and the dimensionally reduced formulation, one finds that the three-form field strengths in the dimensionally reduced effective action of the type IIA theory carry an additional factor of $e^{-\alpha\rho}$. The four-form field strengths, which only exist in $d \geq 8$ space-time dimensions follow the same pattern, with the dimensionally reduced formulation containing an additional factor of $e^{-\alpha\rho}$ when compared to the E_{n+1} formulation of the effective action in d dimensions. Thus, one finds that the surplus weight of any Cartan form or field strength in the dimensionally reduced formulation of the effective action of the type IIA theory in d dimensions contains an additional factor of $e^{-\alpha\rho}$ when compared to the E_{n+1} formulation in d dimensions. The dimensionally reduced theory also carries a factor of $e^{-\tilde{s}\phi}$ from the ten dimensional automorphic form, where $\tilde{s}$ is given in equation (7.12) and is fixed by demanding that, upon transforming to string frame, any term carries a factor of $e^{\phi(2g-2)}$ arising from a perturbative expansion in the ten dimensional IIA string coupling constant $g_s = e^{\phi}$ at order g.

Therefore, we find that the dimensionally reduced theory, when compared to the corresponding product of Cartan forms and field strengths in the E_{n+1} formulation, comes with a surplus factor of

$$e^{-(l_T-2)\alpha\rho-\tilde{s}\phi}. \tag{7.61}$$

where $l_T = \tilde{l}_0 + \tilde{l}_R + \tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 + \tilde{l}_4$ is the total number of derivatives and $\tilde{s}$ is given in equation (7.12). This factor must be attributed to the automorphic form in the E_{n+1} formulation which has leading order contribution $e^{-\sqrt{2}\vec{\Lambda}_\phi\cdot\vec{\phi}}$. Therefore, if we define $\vec{\Lambda}_\phi$ to be the highest weight of the representation out of which the automorphic form Φ_E is constructed in the E_{n+1} formulation, we find

$$\vec{\Lambda}_\phi = \left(\frac{\tilde{s}}{\sqrt{2}}, \alpha\frac{\left(l^T-2\right)}{\sqrt{2}}, \underline{0}\right) = \left(\frac{l_T-2}{4} + \frac{3}{4}\left(l_{RR}-2g\right)\right)\vec{\Lambda}_n + \left(\frac{1}{2}\left(l_{RR}-2g\right)\right)\vec{\Lambda}_{n+1}. \tag{7.62}$$

where $l_{RR} = \tilde{l}_2 + \tilde{l}_4$ is the number of R-R fields in a given term.

For a pure NS-NS term at $g=0$, (i.e. setting l_2=l_4 and $g=0$) the leading order contribution to the automorphic form carries the weight

$$\vec{\Lambda}_\phi = \left(\frac{l_T-2}{4}\right)\Lambda_n. \tag{7.63}$$

Therefore a pure NS-NS higher derivative term in $d = 10-n$ dimensions at $g=0$ should contain an automorphic form with $s = \left(\frac{l_T-2}{4}\right)$.

7.5 Conclusion

In this chapter we have carried out the dimensional reduction of the higher derivative corrections to the effective action of type IIA string theory and found that the E_{n+1} automorphic forms that appear as coefficients of the terms in the effective action in $d = 10 - n$ dimensions must contain the fundamental weight Λ_n associated with node n of the E_{n+1} Dynkin diagram of figure 12; this corresponds to node ten in the E_{11} Dynkin diagram of figure 14. The well understood E_{n+1} automorphic forms that appear in string theory are constructed using a given representation of E_{n+1}; the reader may, for example, consult the explicit construction of these objects given in [40]. As such, the result of this chapter strongly suggests that the automorphic forms that occur in string theory are constructed from the representation with highest weight $\vec{\Lambda}_n$. More precisely, it implies that if the coefficient of the higher derivative term is a sum of automorphic forms then one of them should be constructed from the highest weight $\vec{\Lambda}_n$ as it could happen that the other automorphic forms do not occur in the dimensional reduction from the IIA theory in ten dimensions. A similar analysis from the IIB perspective gave the same result namely that the automorphic form contains the weight $\vec{\Lambda}_n$ [28]. However, from the M theory perspective, that is from eleven dimensions, a similar analysis found that the automorphic form contains the highest weight $\vec{\Lambda}_{n+1}$ which in the E_{11} Dynkin diagram of figure 14 corresponds to node eleven [28]. This result only applies to terms that occur in the eleven dimensional theory. The calculation of this chapter, and that of reference [28] also determines the parameter s of equation (7.16) that occurs in the automorphic form; for the IIA and IIB theories we find that $s = \frac{l_T - 2}{4}$, while for M theory we find that $s = \frac{l_T - 2}{6}$ where l_T is the number of space-time derivatives in the term in the effective action being considered.

We will now consider if the results just mentioned actually agree with the known results in type II string theory. For low numbers of space-time derivatives there are precise proposals for the automorphic forms that occur and their properties have been checked against the known features of the perturbation expansions of the type II strings [40, 45–47]. One finds for the R^4 term in $d \leq 7$ that the E_{n+1} automorphic form is built from the representation with highest weight $\vec{\Lambda}_n$ and has $s = \frac{3}{2}$. This is completely consistent with the results found from the IIA and IIB viewpoints. For the $\partial^4 R^4$, or equivalently R^6, term in $d \leq 7$ the E_{n+1} automorphic form is also built from the representation with highest weight $\vec{\Lambda}_n$ and has $s = \frac{5}{2}$. However, in $d = 7$ dimensions the coefficient of this term is in fact a sum of two $E_4 = SL(5)$ automorphic terms [45–47], in addition to an automorphic form constructed from the $\mathbf{\overline{5}}$ of $SL(5)$, with $s = \frac{5}{2}$ one finds an automorphic form built from the $\mathbf{\overline{10}}$ of $SL(5)$ with $s = \frac{5}{2}$. Similarly, in $d = 6$ dimensions the coefficient of the $\partial^4 R^4$ term is the sum of an automorphic form constructed from the $\mathbf{\overline{10}}$ of $SO(5,5)$, with $s = \frac{5}{2}$ and another automorphic form built from the 16-dimensional representation of $SO(5,5)$ with $s = 3$. As these additional automorphic forms disappear in the limits being considered the known automorphic forms for the $\partial^4 R^4$ term are also consistent with the results found from the dimensional reduction of the type IIA and type IIB theories.

However, dimensional reduction of the higher derivative correction of the eleven dimensional theory [28] suggests that the automorphic forms are constructed from the representation with highest weight $\vec{\Lambda}_{n+1}$. At first sight this is inconsistent with the automorphic forms that are known to be present. However, in seven dimensions, i.e. for $SL(5)$, for the R^4 term this would imply in particular that the automorphic form constructed from the $\overline{\mathbf{5}}$ of $SL(5)$ with $s = \frac{3}{2}$ is proportional to the automorphic form constructed from the $\mathbf{5}$ of $SL(5)$ with $s = 1$. In fact this relation follows from the observation that an automorphic form constructed from a given representation and another automorphic form constructed from the corresponding Cartan involution twisted representation are related by two suitable values of s [40]. The same holds for the automorphic forms associated with the R^4 terms in lower dimensions as one knows [47] that the automorphic form constructed from the representation of E_{n+1} with highest weight $\vec{\Lambda}_n$ and $s = \frac{3}{2}$, i.e $\Phi^{E_{n+1}}_{\Lambda_n;\frac{3}{2}}$ is proportional to the automorphic form constructed from the representation of E_{n+1} with highest weight $\vec{\Lambda}_{n+1}$ and $s = 1$, i.e $\Phi^{E_{n+1}}_{\Lambda_{n+1};1}$, that is [47]

$$\Phi^{E_{n+1}}_{\Lambda_n;\frac{3}{2}} \propto \Phi^{E_{n+1}}_{\Lambda_{n+1};1}. \tag{7.64}$$

Some examples of relationships of this type were also found in reference [37]. Consequently, the known automorphic forms that occur for the R^4 term are also in agreement with the prediction from the M theory viewpoint. However, one can not apply the M theory results to the R^6 term as this term does not occur in the higher derivative effective action in eleven dimensions and so is not included in the analysis from the M theory viewpoint given in [28]. Indeed, the only terms that occur in eleven dimensions that involve, for example, the Riemann curvature are of the form R^{3n+1}, for n a positive integer.

Given the above discussion, it is tempting to suppose the following

- The automorphic forms that occur as coefficients of the higher derivative terms in the string theory effective action must contain an automorphic form constructed from the Λ_n representation of E_{n+1}.

- The automorphic forms that occur in string theory and built from the Λ_n representation of E_{n+1} are the same as the automorphic forms built from the Λ_{n+1} representation of E_{n+1} up to a numerical factor.

The first statement is phrased so as to allow for the possibility that the coefficient is a sum of automorphic forms one or more of which may disappear in the limit. The second statement only applies to automorphic forms of higher derivative terms that occur in eleven dimensions.

The automorphic forms that are used in the recent work of [45–47] are those that appear in the work of Langlands, and they are eigenfunctions of the Laplacian and the higher Casimir operators of E_{n+1}. The automorphic forms constructed in equation (7.16) are not in general eigenfunctions of these operators. However, one can impose constraints on the representations used to construct the automorphic forms and they then do become eigenfunctions of the Laplacian and higher Casimir operators. This has been worked out explicitly for the case of six dimensions, i.e. for $SO(5,5)$

with the ten dimensional vector representation where the constraint is that the length squared of this vector vanish. Indeed only if this constraint is implemented is the perturbation series in agreement with that found in string theory; this part of the automorphic form has been checked in detail to agree with the $SO(5,5)$ Langlands automorphic form for this representation [40]. It remains, however, to carry out the analogue of this construction for the higher rank groups and representations. It is interesting to note that at least the constant part of the Langlands automorphic forms can be written as a sum of the Weyl group and this, being a rotation, preserves the lengths of vectors and those vectors that do occur must belong to a single orbit. As such, it is likely that the Langlands automorphic forms will involve constraints on the representations used and will agree with the automorphic forms of equation (7.16) once one imposes the appropriate constraints.

As we have mentioned, the detailed studies of the automorphic forms in the effective actions of type IIA/B string theory and M-theory have only concerned terms which have low numbers of space-time derivatives. However, it is known that the automorphic forms that occur as coefficients of the higher derivative terms in ten dimensions that have more than twelve space-time derivatives, are not eigenvalues of the Laplacian and so they can not be the Eisenstein automorphic forms found say in the Langlands papers [47]. As a result the automorphic forms that occur for these higher derivative terms are essentially unknown. This chapter and reference [28] puts some constraints on these objects. We have tacitly assumed that all of the automorphic forms that appear as the coefficient functions of the higher derivative terms are constructed from a representation of E_{n+1}. Although the form of equation (7.16) may not be correct in general, even with constraints, the automorphic forms will still have a dominant behaviour of the form $e^{-sw\phi}$ in the limit studied in this chapter, so they will contain a parameter s.

We will now comment on the significance of the representations that occur in the automorphic forms. The brane charges of type II string theory in d-dimensions belong to representations of E_{n+1}. In fact, there is very substantial evidence to believe that all brane charges belong to the l_1 representation of E_{11} [73, 75–77]. Carrying out the decomposition of the l_1 representation we find the brane charges in d dimensions; they are listed in table 3 [75–77]. The first entries of the table agree with that found earlier using U duality transformations [86]. Examining the table we find that the string charges, i.e Z^a, are in the Λ_n representation, the membrane charges, i.e Z^{ab}, are in the Λ_{n+1} representation and the point particle charges, i.e Z, are in the Λ_1 representation. Thus the above propositions can be expressed as

- The automorphic forms that occur as coefficients of the higher derivative terms in the string theory effective action are constructed from the the string charge representation. We may very generically write these automorphic forms as Φ_{string}.

- The automorphic forms that occur in string theory built from the string charge representation are the same as the automorphic forms built from the membrane charge representation, up to a numerical factor. We may very generically write this as $\Phi_{string} = \Phi_{membrane}$.

d	G	Z	Z^a	$Z^{a_1a_2}$	$Z^{a_1a_2a_3}$	$Z^{a_1\dots a_4}$	$Z^{a_1\dots a_5}$	$Z^{a_1\dots a_6}$	$Z^{a_1\dots a_7}$
8	$SL(3)\times SL(2)$	$(\mathbf{3},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	$(\overline{\mathbf{3}},\mathbf{2})$	$(\mathbf{1},\mathbf{3})$ $(\mathbf{8},\mathbf{1})$ $(\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{2})$ $(\mathbf{6},\mathbf{2})$	$(\mathbf{6},\mathbf{1})$ $(\mathbf{18},\mathbf{1})$ $(\mathbf{3},\mathbf{1})$ $(\mathbf{6},\mathbf{1})$ $(\mathbf{3},\mathbf{3})$
7	$SL(5)$	$\mathbf{10}$	$\overline{\mathbf{5}}$	$\overline{\mathbf{5}}$	$\overline{\mathbf{10}}$	$\mathbf{24}$ $\mathbf{1}$	$\mathbf{40}$ $\mathbf{15}$ $\mathbf{10}$	$\mathbf{70}$ $\mathbf{50}$ $\mathbf{45}$ $\mathbf{5}$	- - - -
6	$SO(5,5)$	$\overline{\mathbf{16}}$	$\mathbf{10}$	$\mathbf{16}$	$\mathbf{45}$ $\mathbf{1}$	$\overline{\mathbf{144}}$ $\mathbf{16}$	$\mathbf{320}$ $\mathbf{126}$ $\mathbf{120}$	- - -	- - -
5	E_6	$\overline{\mathbf{16}}$	$\mathbf{27}$	$\mathbf{78}$ $\mathbf{1}$	$\overline{\mathbf{351}}$ $\overline{\mathbf{27}}$	$\mathbf{1728}$ $\mathbf{351}$ $\mathbf{27}$	- - -	- - -	- - -
4	E_7	$\mathbf{56}$	$\mathbf{133}$ $\mathbf{1}$	$\mathbf{912}$ $\mathbf{56}$	$\mathbf{8645}$ $\mathbf{1539}$ $\mathbf{133}$ $\mathbf{1}$	- - - -	- - - -	- - - -	- - - -
3	E_8	$\mathbf{248}$ $\mathbf{1}$	$\mathbf{3875}$ $\mathbf{248}$ $\mathbf{1}$	$\mathbf{147250}$ $\mathbf{30380}$ $\mathbf{3875}$ $\mathbf{248}$ $\mathbf{1}$	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -

Table 3: The brane charge representations of the group G derived from the l_1 representation of E_{11} [75–77]

As before the latter proposition only applies to the terms that have an eleven-dimensional origin. It is of course very natural that the string and membrane charge representations found in the automorphic forms arise from the dimensional reduction of the ten dimensional IIA and IIB string theories and the eleven dimensional theory respectively.

It was also observed in reference [47] that the automorphic form for the R^4 term are related to those built from the Λ_1 representation as follows

$$\Phi^{E_{n+1}}_{\Lambda_n;\frac{3}{2}} \propto \Phi^{E_{n+1}}_{\Lambda_1;\frac{n-2}{2}}\,, \tag{7.65}$$

for $n = 4,5,6,7$ while for the R^6 term

$$\Phi^{E_{n+1}}_{\Lambda_n;\frac{5}{2}} \propto \Phi^{E_{n+1}}_{\Lambda_1;\frac{n+2}{2}}\,, \tag{7.66}$$

for $n = 4,5,6,7$.

Since the charges for the point particle belong to the Λ_1 representation we are also tempted to propose that

- The automorphic forms that occur in string theory are built from the string charge representation are the same as the automorphic forms built from the point charge representation up to a numerical factor. We may generically write this as $\Phi_{string} = \Phi_{point}$.

For the case of $d = 7$ with the group $SL(5)$ this would require that the automorphic forms constructed from the $\overline{\mathbf{5}}$ and $\mathbf{10}$ representations were the same for appropriate representations. In fact the automorphic forms constructed by Langlands for the two representations $\vec{\Lambda}$ and $\vec{\Lambda}'$ are proportional if the vectors $\vec{\lambda} = 2s\vec{\Lambda} - \vec{\rho}$ and $\vec{\lambda}' = 2s'\vec{\Lambda}' - \rho$ are related by a Weyl reflection. The Weyl vector $\vec{\rho}$ can be written as $\vec{\rho} = \sum_a \vec{\Lambda}_a$ where $\vec{\Lambda}_a$ are the fundamental weights. For our case we should take $\vec{\Lambda} = \vec{\Lambda}_3$ and $\vec{\Lambda}' = \vec{\Lambda}_1$. Since Weyl reflections are rotations they preserve the length squared and one finds that $\vec{\lambda}.\vec{\lambda} = \vec{\lambda}'.\vec{\lambda}'$ for $s = \frac{3}{2}$ if $s' = 2$ or $s' = \frac{1}{2}$ and for $s = \frac{5}{2}$ if $s' = \frac{5}{2}$. Indeed one can show that for $s = \frac{3}{2}$ and $s' = \frac{1}{2}$ and also for $s = \frac{5}{2} = s'$ there is a Weyl reflection of the required kind and so the relations of equations (7.65) and (7.66) do extend to the case of $n = 3$ are required. This is most easily found by writing the vectors $\vec{\lambda}$ and $\vec{\lambda}'$ in terms of the orthonormal basis $\vec{e}_a, a = 1, 2, 3, 4, 5$ in terms of which the simple roots take the form $\vec{\alpha}_a = \vec{e}_a - \vec{e}_{a+1}$. As Weyl reflections permute the $\vec{e}_a$ basis it is straightforward to see if the two vectors are related by a Weyl reflection.

8 Construction and Evaluation of an Automorphic Form

This chapter reviews the construction of a class of automorphic forms presented in reference [40]. Although, it is not clear which automorphic forms have a role to play in the effective action of type IIA/B string theory and M-theory, the class of Eisenstein-like automorphic forms we will describe has been argued to appear as the coefficient function of the R^4 and $\partial^4 R^4$ terms in the effective action of type IIB string theory in $d \geq 7$ dimensions [15, 28, 29, 33–36, 39, 40, 45–48]. Suitably constrained versions of these automorphic forms are also expected to be the coefficient functions of the R^4 and $\partial^4 R^4$ terms for $d < 7$ [28, 29, 37, 40, 45–48]. We refer to the class of automorphic forms described in this chapter as Eisenstein-like automorphic forms, since the term Eisenstein series is often reserved for automorphic forms that are Eigenfunctions of the Laplacian on the G/H moduli space, where G is the symmetry group of the automorphic form and H is the maximal compact subgroup of G. Although in some instances Eisenstein-like automorphic forms reduce to Eisenstein series, this is discussed further in section 8.2.

Reference [40] examined the automorphic forms constructed from the representation of $E_{n+1}(\mathbb{R})$ with highest weight $\vec{\Lambda}_{n+1}$, based on the results found in reference [28] and presented in chapter 6, after explaining the construction of this class of automorphic forms we will present an analysis of automorphic forms constructed from several other representations of $E_{n+1}(\mathbb{R})$ not examined in reference [40]. In particular the perturbative parts of the automorphic forms constructed from different representations of $E_{n+1}(\mathbb{R})$ are analysed to investigate the possibility of these automorphic forms appearing as the coefficient function of a higher order term in the effective action of type IIB string theory in $d = 10 - n$ dimensions.

8.1 Construction and Evaluation of Unconstrained Eisenstein-like Automorphic Forms

We begin the construction of our Eisenstein-like automorphic forms by taking a linear representation of $E_{n+1}(\mathbb{Z})$ with highest weight $\vec{\mu}^1$ and dimension N. A state $|\psi >$ in this representation may be expanded in terms of the N states $|\vec{\mu}^i >$ with weight $\vec{\mu}^i$ that provide a basis for the N dimensional representation space. An arbitrary state $|\psi >$ may be written

$$|\psi >= \sum_i m_i |\vec{\mu}^i > , \tag{8.1}$$

where $m_i \in \mathbb{Z}$. The coefficients m_i of the states $|\vec{\mu}^i >$ therefore lie in a lattice Λ defined by

$$\Lambda = \{(m_1, m_2, ..., m_N) : m_i \in \mathbb{Z} \text{ for } i = 1, ..., N \text{ and } m_i \neq 0 \text{ for all } i\} , \tag{8.2}$$

one then has a weight lattice of states of the form $\sum_i m_i|\vec{\mu}^i>$ for $m_i \in \Lambda$. As described in appendix (B.3), any linear representation of a group G carried by a state $|\psi>$ in the representation space of G may be transformed to a state $|\varphi>$ transforming as a non-linear representation of G under the maximal compact subgroup H by taking

$$\begin{aligned} |\varphi> &= L((g(\xi)))^{-1}|\psi> \\ &= \sum_i m_i e^{\frac{1}{\sqrt{2}}\vec{\phi}.\vec{H}} e^{-\sum_{\vec{\alpha}>0}\chi_{\vec{\alpha}}E_{\vec{\alpha}}}|\vec{\mu}^i>, \end{aligned} \tag{8.3}$$

where $g(\xi) \in G/H$, L is a representation of G and we have used the Iwasawa decomposition to express the coset element in terms of a choice of Cartan subalgebra of G with basis elements $\vec{H}$ and a set of positive root generators $E_{\vec{\alpha}}$. From a state $<\psi_{D\tau}|$, expanded in terms of states $<\vec{\mu}_i|$ with weight $\vec{\mu}^i$ and transforming under the Cartan twisted dual representation of G, we may similarly construct a state $<\varphi_{\tau D}|$ transforming as a non-linear representation of G by defining

$$\begin{aligned} <\varphi_{\tau D}| &=<\psi_{D\tau}|L\left(\tau(g(\xi))\right) \\ &= \sum_i <\vec{\mu}_i|m^i e^{-\sum_{\vec{\alpha}>0}\chi_{\vec{\alpha}}E_{-\vec{\alpha}}} e^{\frac{1}{\sqrt{2}}\vec{\phi}.\vec{H}}\,, \end{aligned} \tag{8.4}$$

where τ is the Cartan involution defined in equation (B.42). An invariant automorphic form Φ constructed out of a representation of G with highest weight $\vec{\mu}^1$ is then given by taking

$$\Phi = \sum_{\Lambda} F(u(\xi))\,, \tag{8.5}$$

where Λ is the N dimensional lattice of integers m_i with the origin removed and F is a function of $u(\xi)$, which is defined by,

$$\begin{aligned} u(\xi) &=<\varphi_{\tau D}|\varphi> \\ &=<\psi_{D\tau}|L\left(\tau(g(\xi))\right)|L((g(\xi)))^{-1}|\psi> \\ &= \sum_{i,j} <\vec{\mu}_j|m^j L\left(\tau(g(\xi))\right) L((g(\xi)))^{-1} m_i|\vec{\mu}^i> \\ &= \sum_{i,j} <\vec{\mu}_j|m^j \left(e^{-\sum_{\vec{\alpha}>0}\chi_{\vec{\alpha}}E_{-\vec{\alpha}}} e^{\sqrt{2}\vec{\phi}.\vec{H}} e^{-\sum_{\vec{\alpha}>0}\chi_{\vec{\alpha}}E_{\vec{\alpha}}}\right) m_i|\vec{\mu}^i>\ . \end{aligned} \tag{8.6}$$

The function $u(\xi)$ transforms under G in the following way

$$u(\xi) \to U(g_0)u(\xi) = u(g_0 \cdot \xi')\,. \tag{8.7}$$

Now, the claim is that the automorphic form Φ as a function F of u is invariant under global transformations $U(g_0) \in G(\mathbb{Z})$. To see this, one observes that a $U(g_0)$ transformation of the state

$|\psi>$ contained in u is equivalent to a redefinition of the integer coefficients m_i of the states $|\mu^i>$ in the expansion of $|\psi>$ it then follows that since all lattice states of the form $(m_1, m_2, ..., m_n)$ are summed over in the automorphic form Φ we must have that Φ itself is invariant under a $U(g_0)$ transformation. The case we are concerned with in this section is that where the function F is of the form

$$F(u) = u^{-s}, \tag{8.8}$$

where $s \in \mathbb{R}$. The automorphic form Φ is then

$$\Phi = \sum_{\Lambda} \frac{1}{(u(\xi))^s}. \tag{8.9}$$

The sum over the lattice Λ is convergent for $s > \frac{N}{2}$, the reader is referred to reference [40] for further discussion on the convergence properties of Φ.

So, we have an automorphic form Φ constructed out of a representation of a group G. Clearly, we will take $G = E_{n+1}(\mathbb{R})$ then H is the Cartan involution invariant subgroup of $E_{n+1}(\mathbb{R})$ and the automorphic form Φ is invariant under transformations by group elements in $E_{n+1}(\mathbb{Z})$. However, one has to work a little harder to extract the relevant information from this automorphic form. Specifically, we are interested in splitting the automorphic form into its perturbative and non-perturbative parts.

Examining the non-linearly realised state $|\varphi>$ constructed from some representation of $E_{n+1}(\mathbb{R})$ with highest weight $\vec{\mu}^1$, we have

$$|\varphi>= \sum_i m_i e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{H}} e^{-\sum_{\vec{\alpha}} \chi_{\vec{\alpha}} E_{\vec{\alpha}}} |\vec{\mu}^i>, \tag{8.10}$$

where $\vec{H} = (H_1, H_2, ..., H_{n+1})$ is an $n+1$ vector containing the basis elements H_i of the Cartan subalgebra $E_{n+1}(\mathbb{R})$ in the representation with highest weight $\vec{\mu}^1$ and $E_{\vec{\alpha}}$ are the corresponding positive root generators. The action of the positive root generators on the states $|\vec{\mu}^i>$ is given in appendix $B.2.5$, one has

$$L(E_{\vec{\alpha}})|\vec{\mu}^i>= c_{\vec{\alpha}i}|\vec{\mu}^i + \vec{\alpha}>, \tag{8.11}$$

for some constant $c_{\vec{\alpha}i}$. Therefore the action of the positive root generator part of the group element $L(g)$ on the states $|\vec{\mu}^i>$ is

$$L(e^{-\sum_{\vec{\alpha}} \chi_{\vec{\alpha}} E_{\vec{\alpha}}}) = |\vec{\mu}^i> - \sum_j \tilde{\chi}_{ij} |\vec{\mu}^j>, \tag{8.12}$$

where $\tilde{\chi}_{ij}$ are polynomials in the fields χ_i. The fields $\tilde{\chi}_{ij}$ may be evaluated explicitly by taking the inner product of (8.12) with a suitably normalised state $<\vec{\mu}_j| \left(\omega^j\right)^{-1}$, with $\left(\omega^j\right)^{-1}$ playing the role of the normalising factor, one then has

$$\begin{aligned}\tilde{\chi}_{ki} &= \delta_{ki} - \omega_k^{-1} < \vec{\mu}^k | L(e^{-\sum_{\vec{\alpha}} \chi_{\vec{\alpha}} E_{\vec{\alpha}}}) | \vec{\mu}^i > \\ &= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!\omega_k} \sum_{\vec{\alpha}_1} \cdots \sum_{\vec{\alpha}_n} \chi_{\vec{\alpha}_1} \cdots \chi_{\vec{\alpha}_n} < \vec{\mu}^k | L(E_{\vec{\alpha}_1} \ldots E_{\vec{\alpha}_n}) | \vec{\mu}^i > \\ &= c_{\vec{\alpha}_{ki} i} \chi_{\vec{\alpha}_{ki}} + Poly(\chi_{\vec{\beta}}, 0 < \vec{\beta} < \vec{\alpha}_{ki}),\end{aligned} \tag{8.13}$$

where the $\vec{\alpha}_{ki}$ is a sum of simple roots satisfying $\vec{\alpha}_{ki} = \vec{\mu}^k - \vec{\mu}^i$ and $Poly$ is a polynomial in the fields $\chi_{\vec{\beta}}$ that appear as the parameter for positive root generators E_β such that $0 < \vec{\beta} < \vec{\alpha}_{ki}$. Thus one finds that the non-linearly realised lattice state $|\varphi>$ can be written

$$\begin{aligned}|\varphi> &= \sum_i m_i L(e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{H}}) \left(|\vec{\mu}^i> - \sum_k \tilde{\chi}_{ki} |\vec{\mu}^k> \right) \\ &= \sum_i m_i \left(e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^i} |\vec{\mu}^i> - \sum_k \tilde{\chi}_{ki} e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^k} |\vec{\mu}^k> \right) \\ &= \sum_i \left(m_i - \sum_{j>i} \tilde{\chi}_{ij} m_j \right) e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^i} |\vec{\mu}^i>,\end{aligned} \tag{8.14}$$

where in the last line the sum over i has been rearranged. Therefore, in terms of the fields $\vec{\phi}$ and $\chi_{\vec{\alpha}}$ parameterising the coset $g \in E_{n+1}(\mathbb{R})/H$ the automorphic form Φ is given by

$$u = \sum_{i=1}^{N} (m_i - \tilde{\chi}_i)^2 \omega_i e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^i}, \tag{8.15}$$

where the fields $\chi_{\vec{\alpha}}$ that act as the parameters for the positive root generators $E_{\vec{\alpha}}$ feature in this expression purely through $\tilde{\chi}_i$ which is defined by

$$\tilde{\chi}_i = \sum_{j>i} \tilde{\chi}_{ij} m^j. \tag{8.16}$$

One may observe that $\tilde{\chi}_j$ only contains lattice points m_k that appear as the coefficients of states $|\vec{\mu}^k>$ lower in the weight string than $\vec{\mu}^j$, i.e., $\tilde{\chi}_j$ does not depend on lattice points m_k such that $j < k$. Clearly one of the constants ω_i and $c_{\vec{\alpha}i}$ in u may be fixed by scaling the states of the representation of $E_{n+1}(\mathbb{R})$. We will take the convention of [40] and choose $\omega_i = 1$, the constants $c_{\vec{\alpha}i}$ may then be determined through the Lie algebra relations.

Returning to the automorphic form Φ constructed from u in (8.15) and rewriting Φ using equation $(A.2)$ one has

$$\Phi = \sum_{\Lambda} \frac{1}{u^s} = \sum_{\Lambda} \frac{\pi^s}{\Gamma(s)} \int_0^{\infty} \frac{dt}{t^{1+s}} e^{-\frac{\pi}{t}u}. \tag{8.17}$$

The sum over the lattice Λ may then be split into a sum over $m_1 \neq 0$ and all other $m_k = 0$ and another sum over m_1 including zero, in addition to all m_k, for $k > 1$, except the lattice points $m_k = 0$, for all $k = 2, 3, ...N$, this gives

$$\begin{aligned}\Phi_s = &\sum_{m_1 \neq 0} \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} e^{-\frac{\pi}{t} m_1^2 \omega_1 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^1}} \\ &+ \sum_{m_1=-\infty}^{\infty} \sum_{\Lambda_1} \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} e^{-\frac{\pi}{t}(m_1 - \tilde{\chi}_1)^2 \omega_1 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^1}} e^{-\frac{\pi}{t} u_1},\end{aligned} \tag{8.18}$$

where Λ_1 is the $N-1$ dimensional lattice spanned by $m_2, ..., m_N \in \mathbb{Z}$, excluding the origin and the function u_1 is

$$u_1 = \sum_{i>1} (m_i - \tilde{\chi}_i)^2 \omega_i e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^i}. \tag{8.19}$$

The first sum in (8.18) may be evaluated with equations $(A.2)$ and $(A.3)$, while in the second sum, performing a Poisson resummation over m_1 as given in equation $(A.1)$, leads to

$$\begin{aligned}\Phi_s = &\, 2\zeta(2s) \frac{1}{\omega_1^s} e^{-\sqrt{2}s\vec{\phi}\cdot\vec{\mu}^1} \\ &+ \sum_{\hat{m}_1=-\infty}^{\infty} \sum_{\Lambda_1} \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} \sqrt{\frac{t}{\omega_1}} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1} e^{2\pi i \hat{m}_1 \tilde{\chi}_1} \\ &\qquad \times e^{-\pi t \hat{m}_1^2 \omega_1^{-1} e^{-\sqrt{2}\vec{\phi}\cdot\vec{\mu}^1}} e^{-\frac{\pi}{t} u_1}.\end{aligned} \tag{8.20}$$

It is now possible to write down a recursion relation by defining $\Phi_s = \Phi_s^N$ and splitting the summation over $\hat{m}_1$ and the lattice Λ_1 into a sum over Λ_1 with $\hat{m}_1 = 0$ and another sum over Λ_1 and over all $\hat{m}_1 \neq 0$, this gives

$$\begin{aligned}\Phi_s^N &= 2\zeta(2s) \frac{1}{\omega_1^s} e^{-\sqrt{2}s\vec{\phi}\cdot\vec{\mu}^1} \\ &+ \sum_{\Lambda_1} \sqrt{\frac{\pi}{\omega_1}} \frac{\Gamma(s-1/2)}{\Gamma(s)} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1} \frac{\pi^{s-1/2}}{\Gamma(s-1/2)} \int_0^\infty \frac{dt}{t^{1+(s-1/2)}} e^{-\frac{\pi}{t} u_1} \\ &+ \frac{2}{\sqrt{\omega_1}} \sum_{\hat{m}_1=1}^{\infty} \sum_{\Lambda_1} \cos(2\pi \hat{m}_1 \tilde{\chi}_1) \frac{\pi^s}{\Gamma(s)} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1} \int_0^\infty \frac{dt}{t^{1+(s-1/2)}} e^{-\pi t \hat{m}_1^2 \omega_1^{-1} e^{-\sqrt{2}\vec{\phi}\cdot\vec{\mu}^1}} e^{-\frac{\pi}{t} u_1} \\ &= 2\zeta(2s) \frac{1}{\omega_1^s} e^{-\sqrt{2}s\vec{\phi}\cdot\vec{\mu}^1} + \sqrt{\frac{\pi}{\omega_1}} \frac{\Gamma(s-1/2)}{\Gamma(s)} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1} \Phi_{s-1/2}^{N-1} + \Upsilon_s^N.\end{aligned} \tag{8.21}$$

The automorphic form $\Phi_{s-1/2}^{N-1}$ on the right hand side of the above equation is of identical form to Φ_s^N except that it is constructed from the $N-1$ states $|\vec{\mu}^k >$, $k = 2, ..., N$ and the value of s is shifted. In addition, the Bessel function integral formula $(A.4)$ may be used to express the function

Υ_s^N in the form

$$\Upsilon_s^N = \frac{4}{(\sqrt{\omega_1})^{s+1/2}} \frac{\pi^s}{\Gamma(s)} e^{-\frac{1}{\sqrt{2}}(s+1/2)\vec{\phi}\cdot\vec{\mu}^1} \sum_{\hat{m}_1=1}^{\infty} \sum_{\Lambda_1} \left(\frac{\hat{m}_1}{\sqrt{u_1}}\right)^{s-1/2} \cos(2\pi \hat{m}_1 \tilde{\chi}_1) \times K_{s-1/2}\left(\frac{2\pi}{\sqrt{\omega_1}} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1} \hat{m}_1 \sqrt{u_1}\right). \tag{8.22}$$

Taking the split

$$\Phi_s^N = \Phi_{s\ p}^N + \Phi_{s\ np}^N, \tag{8.23}$$

where $\Phi_{s\ p}^N$ is the perturbative part of the automorphic form and $\Phi_{s\ np}^N$ is the non-perturbative part. We see that

$$\Phi_{s\ np}^N = \Upsilon_s^N, \tag{8.24}$$

since $K_{s-1/2}(x) \sim e^{-x}$ is exponentially suppressed in the large x limit. It then follows that the perturbative part of the automorphic form is given by

$$\Phi_{s\ p}^N = 2\zeta(2s)\frac{1}{\omega_1^s} e^{-\sqrt{2}s\vec{\phi}\cdot\vec{\mu}^1} + \sqrt{\frac{\pi}{\omega_1}} \frac{\Gamma(s-1/2)}{\Gamma(s)} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1} \Phi_{s-1/2 p}^{N-1}. \tag{8.25}$$

Iterating the perturbative and non-perturbative parts of the automorphic form N times gives

$$\Phi_p = \sum_{k=1}^{N} \frac{2}{\omega_k^{s-\frac{k-1}{2}}} \zeta(2s-k+1) \pi^{\frac{k-1}{2}} \frac{\Gamma(s-\frac{k-1}{2})}{\Gamma(s)} e^{-\sqrt{2}(s-\frac{k-1}{2})\vec{\phi}\cdot\vec{\mu}^k} \prod_{i<k} \sqrt{\frac{1}{\omega_i}} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^i} \tag{8.26}$$

and

$$\Phi_{np} = \sum_{k=1}^{N-1} \frac{4}{(\sqrt{\omega_k})^{s-\frac{k-2}{2}}} \frac{\pi^s}{\Gamma(s)} e^{-\frac{1}{\sqrt{2}}(s-\frac{k-2}{2})\vec{\phi}\cdot\vec{\mu}^k} \prod_{j<k} \sqrt{\frac{1}{\omega_j}} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^j} \cos(2\pi \hat{m}_k \tilde{\chi}_k) \times \sum_{\hat{m}_k=1}^{\infty} \sum_{\Lambda_k} \left(\frac{\hat{m}_k}{\sqrt{u_k}}\right)^{s-\frac{k}{2}} K_{s-\frac{k}{2}}\left(\frac{2\pi}{\sqrt{\omega_k}} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^k} \hat{m}_k \sqrt{u_k})\right), \tag{8.27}$$

where Φ_p and Φ_{np} are the complete perturbative and non-perturbative parts of the automorphic form, respectively, and one may write

$$\Phi = \Phi_p + \Phi_{np}\,. \tag{8.28}$$

Note that the perturbative part Φ_p of the automorphic form Φ is independent of the fields $\chi_{\vec{\alpha}}$ and is therefore, in a sense, completely perturbative. In chapter nine we will examine an alternative definition of the perturbative part of the automorphic form defined by taking the weak coupling limit of the d dimensional effective coupling g_d that retains some dependence on the axionic fields $\chi_{\vec{\alpha}}$.

8.2 Constrained Eisenstein-like Automorphic Forms

The coefficient functions for the R^4, $\partial^4 R^4$ and $\partial^6 R^4$ terms in the effective action of type IIB string theory in $d = 10 - n$ dimensions are constrained by supersymmetry to satisfy a set of Poisson equations on the E_{n+1}/H moduli space [45,46] which are given by,

$$\left(\Delta_d - \frac{3\,(11-d)\,(d-8)}{d-2}\right) Z^{(d)}_{R^4} = 6\pi\delta_{d-8,0} \tag{8.29}$$

$$\left(\Delta_d - \frac{5\,(12-d)\,(d-7)}{d-2}\right) Z^{(d)}_{\partial^4 R^4} = 40\zeta\,(2)\,\pi\delta_{d-7,0} \tag{8.30}$$

$$\left(\Delta_d - \frac{6\,(14-d)\,(d-6)}{d-2}\right) Z^{(d)}_{\partial^6 R^4} = -\left(Z^{(d)}_{R^4}\right)^2 + 120\zeta\,(3)\,\pi\delta_{d-6,0} \tag{8.31}$$

where Δ_d is the Laplacian on the E_{n+1}/H moduli space, $Z_{(d)}$ is the d dimensional coefficient function of the higher derivative term indicated in the subscript and ζ is the Riemann zeta function.

In general the Eisenstein-like automorphic forms constructed in section 8.1 are not eigenfunctions of the Laplacian and do not satisfy the Poisson equations of the coefficient functions of the R^4, $\partial^4 R^4$ and $\partial^6 R^4$ terms. The coefficient function of the R^4 term in ten dimensions is the $SL(2)$ Eisenstein series (1.2) which may be constructed in the manner described in the previous section and does indeed satisfy the required Laplace equation on the $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ moduli space. The coefficient functions of the R^4 terms in $d = 7$, $d = 8$ and $d = 9$ dimensions may similarly be constructed in this way, although in $d = 8$ and $d = 9$ dimensions a sum of E_{n+1} automorphic forms are required to satisfy the Laplace equation on the E_{n+1}/H moduli space. However, the coefficient function of the R^4 term in $d < 7$ dimensions satisfies a Poisson equation that is not immediately satisfied by the Eisenstein-like automorphic form described in the previous section. Instead, in $d = 6$ dimensions, the Eisenstein-like $SO(5,5)$ automorphic form satisfies the Poisson equation of the R^4 coefficient function on the $SO(5,5)$ moduli space only once one implements a constraint on the $SO(5,5)$ lattice, as first described in reference [37] and evaluated in [40]. It is believed that similar constraints must be imposed for the Eisenstein-like E_{n+1} automorphic forms to satisfy the required Poisson equations of the coefficient function of the R^4 term on the E_{n+1}/H moduli space, but as yet the coefficient functions of the R^4 terms in $d < 6$ dimensions have not been constructed and evaluated in this way. Eisenstein-like automorphic forms require similar constraints to be the coefficient functions of the $\partial^4 R^4$ term in $d \leq 7$ dimensions. Different constraints are likely to be necessary for the Eisenstein-like automorphic forms to appear as the coefficient function of the $\partial^6 R^4$ term which satisfies a more complicated Poisson equation.

The coefficient functions of the R^4, $\partial^4 R^4$ and $\partial^6 R^4$ terms given in references [45–47] that satisfy the required Poisson equations are built from the Langlands construction of an automorphic form. The Langlands construction of an automorphic form and the construction of an automorphic form given in section (8.1) are known to give rise to equivalent automorphic forms in certain instances and it is thought that, with suitable constraints, Eisenstein-like automorphic forms will be equivalent to the Langlands automorphic forms appearing as the coefficient functions of the R^4, $\partial^4 R^4$ and $\partial^6 R^4$ terms in references [45–47] .

The terms in the effective action of type IIB string theory in $d = 10 - n$ dimensions at higher orders than the $\partial^6 R^4$ term are not expected to be protected by supersymmetry [46] and little is known about the coefficient functions of any of these higher order terms, although reference [42] makes conjectures for the Poisson equation satisfied by the $\partial^{10} R^4$ term in $d = 10$ dimensions. Since, in general, the constraints on the coefficient functions of the higher derivative terms beyond the $\partial^6 R^4$ term are unknown it is possible that unconstrained Eisenstein-like automorphic forms will appear as the coefficient functions of these terms, the rest of this chapter studies this possibility for unconstrained Eisenstein-like automorphic forms constructed from several representations of E_{n+1} less often considered in the literature.

8.3 Conditions on the Perturbative Parts of the Automorphic Form

To investigate the role of automorphic forms constructed from representations of $E_{n+1}(\mathbb{R})$ in the method outlined in the previous section we will examine the suitability of the perturbative part of the automorphic form as a coefficient function for an arbitrary higher derivative correction in the type IIB effective action in $d = 10 - n$ dimensions. The perturbative part Φ_p of the automorphic form Φ constructed from a representation of $E_{n+1}(\mathbb{R})$ is expressed in terms of the fields $\vec{\phi}$ parameterising the Cartan subalgebra of the coset $g \in E_{n+1}(\mathbb{R})/H$. To deduce whether the perturbative part of the automorphic form gives an acceptable coefficient function for a higher derivative correction to the type IIB effective action in $d = 10 - n$ dimensions we must instead express Φ_p in terms of the type IIB string theory parameters in $d = 10 - n$ dimensions, namely the effective coupling g_d and the volume of the n torus V_n.

The contributions to the effective action of type IIB string theory from a perturbative expansion in $g_s = e^{\phi}$ takes the schematic form

$$S = \frac{1}{\alpha'^4} \int d^{10}x \sqrt{-g_S} g_s^{-2} R_S + ..., \tag{8.32}$$

where the subscript S is to notify the reader that we have written the action in string frame. Compactifying the effective action on an n torus through our standard ansatz (3.11) one finds

$$\begin{aligned} S &= \frac{1}{\alpha'^4} \int d^d x \sqrt{-g_S} V_n g_s^{-2} R_S + ... \\ &= \frac{1}{\alpha'^{\frac{d-2}{2}}} \int d^d x \sqrt{-g_S} g_d^{-2} R_S + ... \, , \end{aligned} \tag{8.33}$$

where the d dimensional effective coupling g_d is given by $g_d = \alpha'^{\frac{n}{4}} g_s V_n^{-\frac{1}{2}}$ and V_n is the volume of the n torus in string frame. Comparing with our compactification ansatz we see that the volume of the n torus V_n in string frame and the $d = 10 - n$ dimensional effective coupling g_d may be expressed in terms of the moduli of the n torus and the type IIB dilaton as

$$\begin{aligned} V_n &= e^{n\phi/4 + n\beta\rho}, \\ g_d &= e^{\frac{8-n}{8}\phi - n\beta\rho/2} . \end{aligned} \tag{8.34}$$

The terms beyond two derivatives in the effective action of type IIB string theory in $d = 10 - n$ dimensions are polynomials in the d dimensional curvature R, $E_{n+1}(\mathbb{R})$ Cartan forms $\mathcal{S}$ and non-linearly realised field strengths $\mathcal{F}$ with a coefficient function that transforms as an $E_{n+1}(\mathbb{Z})$ automorphic form. Therefore, an arbitrary higher order term in the $d = 10 - n$ dimensional effective action of type IIB string theory in Einstein frame has the schematic form

$$l_d^{l-d} \int d^d x \sqrt{-g} \Phi_{E_{n+1}} \mathcal{O}, \tag{8.35}$$

where l_d is the d dimensional Planck length, $\Phi_{E_{n+1}}$ is an $E_{n+1}(\mathbb{Z})$ automorphic form and $\mathcal{O}$ is a polynomial in the d dimensional curvature R, $E_{n+1}(\mathbb{R})$ Cartan forms $\mathcal{S}$ and non-linearly realised field strengths $\mathcal{F}$, while l is the number of derivatives in $\mathcal{O}$. Upon rescaling to $d = 10 - n$ dimensional string frame the perturbative part of an arbitrary higher derivative term in (8.35) must agree with a perturbative expansion in the effective coupling g_d, that is, each term must have a dependence on the effective coupling of the form g_d^{-2+2g}, where $g = 0, 1, 2, ...$ is the genus. The d dimensional Einstein frame metric is related to the d dimensional string frame metric by

$$g_{\mu\nu} = g_d^{-\frac{4}{d-2}} g_{(S)\mu\nu}, \tag{8.36}$$

where $g_{\mu\nu}$ and $g_{(S)\mu\nu}$ are the components of the d dimensional metric in Einstein and string frame, respectively. Thus, we find that an arbitrary higher derivative term in the $d = 10 - n$ dimensional effective action of type IIB string theory in string frame is

$$l_s^{l-d} \int d^d x \sqrt{-g_S} g_d^{\frac{4\Delta - 2d}{d-2}} \Phi_{E_{n+1}} \mathcal{O}_S, \tag{8.37}$$

where Δ is the number of space time metrics that transform as a contravariant tensor minus the number of space time metrics that transform as a covariant tensor in $\mathcal{O}_S$. It is then apparent that to agree with terms arising from a perturbative expansion in g_d the product

$$g_d^{\frac{4\Delta - 2d}{d-2}} \Phi_{E_{n+1}\ p} \mathcal{O}_S \tag{8.38}$$

can only be comprised of terms with a dependence on the effective coupling of the form g_d^{2g-2}, where each term may arise from a different genus g.

8.4 Extracting the Perturbative Part of the Automorphic Form

The calculation of the perturbative part of an automorphic form constructed from a representation of $E_{n+1}(\mathbb{R})$ with highest weight $\vec{\Lambda}_k$ essentially involves evaluating the inner products of a vector of fields $\vec{\phi}$ that parameterise the Cartan subalgebra of $E_{n+1}(\mathbb{R})$ and the weights $\vec{\mu}^i$ in the root string of $\vec{\Lambda}_k$.

To proceed with this calculation we will decompose the $E_{n+1}(\mathbb{R})$ representation into representations of $GL(1,\mathbb{R}) \times SL(2,\mathbb{R}) \times SL(n,\mathbb{R})$ by deleting node n of the E_{n+1} Dynkin diagram given in figure 16. Under this decomposition the simple roots and fundamental weights of $E_{n+1}(\mathbb{R})$ de-

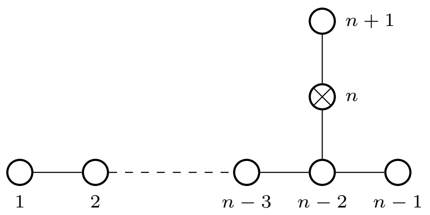

Figure 16: Dynkin diagram for E_{n+1} with type IIB labeling

compose into the simple roots and fundamental weights of $GL(1,\mathbb{R}) \times SL(2,\mathbb{R}) \times SL(n,\mathbb{R})$, the derivation of this decomposition is given in appendix $D.3$. We then find that the simple roots $\vec{\alpha}_k$ and fundamental weights $\vec{\Lambda}_k$ of $E_{n+1}(\mathbb{R})$ may be written

$$\vec{\alpha}_{n+1} = (\beta_1, 0, 0), \qquad \vec{\alpha}_n = (0, x, \underline{0}) - \vec{\nu}, \qquad \vec{\alpha}_i = (0, 0, \underline{\alpha}_i) \tag{8.39}$$

and

$$\vec{\Lambda}^i = \left(0, \frac{1}{x}\underline{\lambda}^{n-2} \cdot \underline{\lambda}^i, \underline{\lambda}^i\right), \qquad \vec{\Lambda}^n = \left(0, \frac{1}{x}, \underline{0}\right), \qquad \vec{\Lambda}^{n+1} = \left(\mu_1, \frac{1}{2x}, \underline{0}\right), \tag{8.40}$$

where $\vec{\nu} = (\mu_1, 0, \underline{0}) + (0, 0, \underline{\lambda}^{n-2})$, the constant $x = \sqrt{\frac{8-n}{2n}}$ is fixed by demanding $\vec{\alpha}_n^2 = 2$, the simple roots and fundamental weights of $SL(n)$ are $\underline{\alpha}_i$ and $\underline{\lambda}^i$, while the simple root β_1 and fundamental weight μ_1 of $SL(2)$ are $\beta_1 = \sqrt{2}$, $\mu_1 = \frac{1}{\sqrt{2}}$. Any root of $E_{n+1}(\mathbb{R})$ can then be expressed as

$$\vec{\alpha} = n_c \vec{\alpha}_n + m\vec{\beta} + \sum_i n_i \vec{\alpha}_i = n_c(0, x, 0) - \vec{\lambda}, \tag{8.41}$$

where $\vec{\lambda} = n_c \vec{\nu} - \sum_i n_i (0, 0, \underline{\alpha}_i) - m(\beta_1, 0, \underline{0})$ is a weight of $SL(2) \otimes SL(n)$ and the integer n_c is the level with respect to the decomposition.

The fields that act as the parameters of the Cartan subalgebra of $E_{n+1}(\mathbb{R})$ in the $d = 10 - n$ dimensional type IIB theory are ϕ, ρ and $\underline{\phi}$, where ϕ is the type IIB dilaton, ρ controls the volume of the n torus and $\underline{\phi}$ is a vector of fields parameterising the remaining $SL(n,\mathbb{R})$ symmetry of the torus. Under the decomposition of $E_{n+1}(\mathbb{R})$ into representations of $GL(1) \times SL(2) \times SL(n)$, we find that the $n+1$ vector $\vec{\phi}$ that appears in perturbative part of the automorphic form is

$$\vec{\phi} = \left(\phi, \rho, \underline{\phi}\right). \tag{8.42}$$

Before evaluating the perturbative part of an automorphic form constructed from a representation of $E_{n+1}(\mathbb{R})$ with highest weight $\vec{\Lambda}_k$ one may define an ordering of the weights. We will follow the convention of [40] and take $\vec{\mu}_i > \vec{\mu}_j$ if the first non-zero component of $\vec{\mu}_i - \vec{\mu}_j$ in the ordered basis $(0, 1, \underline{0})$, $(1, 0, \underline{0})$ and $(0, 0, \underline{\mu})$ is positive. With this definition, the weights are ordered in terms of increasing level n_c followed by their $SL(2)$ weights then with respect to their $SL(n)$ weights. One then finds that the perturbative part of the automorphic form constructed from a representation of $E_{n+1}(\mathbb{R})$ with highest weight $\vec{\lambda}_k$ is given by

$$\Phi_p = \sum_{n_c, i} \sum_{k=a_\alpha+1}^{a_{\alpha+1}} E_k N_\alpha(\phi) e^{-\frac{s}{\sqrt{2}}\vec{\phi}.\vec{\lambda}_k} e^{(2n_c s - n_c a_\alpha + b_\alpha)\frac{x}{\sqrt{2}}\rho} P_k(\underline{\lambda}, \underline{\phi}), \tag{8.43}$$

where $N_\alpha = 1$ if α corresponds to the singlet representation of $SL(2)$, $N_\alpha = e^{-\phi(s-\frac{a_\alpha}{2})}$ if the $SL(2)$ weight is μ_1 and $N_\alpha = e^{\phi(s-\frac{a_\alpha+1}{2})}$ if the $SL(2)$ weight is $-\mu_1$ for the doublet representation of $SL(2)$ and $N_\alpha = e^{-\phi(2s-a_\alpha)}$ if the $SL(2)$ weight is $2\mu_1$, $N_\alpha = e^{-\phi d_{\alpha-1}}$ if the $SL(2)$ weight is 0, $N_\alpha = e^{\phi(2s-a_{\alpha+1})}$ if the $SL(2)$ weight is $-2\mu_1$ for the triplet representation of $SL(2)$. In addition, the functions E_k and P_k are given by

$$E_k = 2\pi^{\frac{k-1}{2}}\zeta(2s-k+1)\Gamma(s-\frac{k-1}{2})/\Gamma(s), \tag{8.44}$$

and

$$P_k(\underline{\lambda}, \underline{\phi}) = e^{-\sqrt{2}((s-\frac{k-1}{2})[\underline{\lambda}]_{k-a_\alpha} + \frac{1}{2}([\underline{\lambda}]_1 + \ldots + [\underline{\lambda}]_{k-1-a_\alpha}))\cdot\underline{\phi}}, \tag{8.45}$$

where $[\underline{\lambda}]_r$ is the r-th weight in the root string with highest $SL(n)$ weight $\underline{\lambda}$. Finally, to compare the perturbative part of the automorphic form to a perturbative expansion in g_d one may convert the fields ϕ and ρ into the physical parameters g_d and V_n through the relations

$$e^{\phi} = g_d V_n^{\frac{1}{2}}, \qquad e^{\frac{\rho}{\sqrt{2}x}} = V_n^{-\frac{1}{2}} g_d^{\frac{n}{8-n}}, \qquad e^{\frac{x}{\sqrt{2}}\rho} = g_d^{\frac{1}{2}} V_n^{-\frac{8-n}{4n}}. \tag{8.46}$$

8.5 Analysis of the Perturbative Part of Automorphic Forms constructed from Various Representations

The automorphic forms considered in reference [40] were constructed exclusively from the representation of $E_{n+1}(\mathbb{R})$ with highest weight $\vec{\Lambda}_{n+1}$, however, it is known that automorphic forms constructed from representations other than those with highest weight $\vec{\Lambda}_{n+1}$ can appear in string theory [45–47]. Let us now examine several of these other representations of $E_{n+1}(\mathbb{R})$ through the construction of the automorphic form reviewed in this section. Details of the derivations of each of these automorphic forms is given in appendix C.

8.5.1 10 of $SL(5)$

The unconstrained Eisenstein-like automorphic form constructed from the representation of $SL(5)$ with highest weight $\vec{\Lambda}^1$ is given by

$$\begin{aligned}\Phi_p =& \sum_{k=1}^{3} V_n^{\frac{2s}{3}} g_d^{-\frac{4}{5}s} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=4}^{6} V_n^{-\frac{2}{3}s+2} g_d^{-\frac{4}{5}s} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) \\ &+ \sum_{k=7}^{9} V_n^{\frac{1}{3}s-1} g_d^{\frac{6}{5}s-6} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) + V_n^{-s+5} g_d^{\frac{6}{5}s-6} E_{10}.\end{aligned} \tag{8.47}$$

Demanding that the first term appears at some minimum order in the perturbative expansion, with associated genus g_0, gives,

$$\frac{4\Delta - 2d}{d-2} - \frac{4}{5}s = -2 + 2g_0. \tag{8.48}$$

For $d = 7$

$$s = \Delta - 1 - \frac{5}{2}g_0. \tag{8.49}$$

So, we find

$$\begin{aligned}g_d^{\frac{4}{5}s-2+2g_0}\phi_p =& \sum_{k=1}^{3} V_n^{\frac{2s}{3}} g_d^{-2+2g_0} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=4}^{6} V_n^{-\frac{2}{3}s+2} g_d^{-2+2g_0} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) \\ &+ \sum_{k=7}^{9} V_n^{\frac{1}{3}s-1} g_d^{2s-8+2g_0} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) + V_n^{-s+5} g_d^{2s-8+2g_0} E_{10}.\end{aligned} \tag{8.50}$$

Comparing the power of g_d in terms 7 to 9 and 10 with g_d^{2g-2}

$$\begin{aligned}g &= s - 3 + g_0 \\ &= \Delta - 4 - \frac{3}{2}g_0.\end{aligned} \tag{8.51}$$

Therefore the perturbative part of the automorphic form constructed from the **10** of $SL(5)$ with highest weight $\vec{\Lambda}_1$ appears to be compatible with any higher derivative term in the $d = 10 - n$ dimensional type IIB string theory effective action such that $\Delta - 4 - \frac{3}{2}g_0$ is a non-negative integer.

8.5.2 16 of $SO(5,5)$

The unconstrained Eisenstein-like automorphic form constructed from the representation of $SO(5,5)$ with highest weight $\vec{\Lambda}^1$ is given by

$$\begin{aligned}\Phi_p =& \sum_{k=1}^{4} V_n^{\frac{s}{2}} g_d^{-s} E_k P_k\left(\underline{\lambda}_1,\underline{\phi}\right) + \sum_{k=5}^{8} V_n^{\frac{s}{2}} g_d^{-s} g_d^{\frac{2s-4}{2}} V_n^{-\frac{(2s-4)}{4}} V_n^{-\frac{s-2}{2}} g_d^{-(s-2)} E_k P_k\left(\underline{\lambda}_3,\underline{\phi}\right) \\ &+ \sum_{k=9}^{12} V_n^{\frac{s}{2}} g_d^{-s} g_d^{\frac{2s-4}{2}} V_n^{-\frac{(2s-4)}{4}} V_n^{\frac{s-6}{2}} g_d^{(s-6)} E_k P_k\left(\underline{\lambda}_3,\underline{\phi}\right) + \sum_{k=13}^{16} V_n^{\frac{s}{2}} g_d^{-s} g_d^{\frac{4s-16}{2}} V_n^{-\frac{(4s-16)}{4}} E_k P_k\left(\underline{\lambda}_1,\underline{\phi}\right)\end{aligned} \tag{8.52}$$

$$\begin{aligned}=& \sum_{k=1}^{4} V_n^{\frac{s}{2}} g_d^{-s} E_k P_k\left(\underline{\lambda}_1,\underline{\phi}\right) + \sum_{k=5}^{8} V_n^{-\frac{s}{2}+2} g_d^{-s} E_k P_k\left(\underline{\lambda}_3,\underline{\phi}\right) \sum_{k=9}^{12} V_n^{\frac{s}{2}-2} g_d^{s-8} E_k P_k\left(\underline{\lambda}_3,\underline{\phi}\right) \\ &+ \sum_{k=13}^{16} V_n^{-\frac{s}{2}+4} g_d^{s-8} E_k P_k\left(\underline{\lambda}_1,\underline{\phi}\right).\end{aligned} \tag{8.53}$$

Demanding that the first term appears at some minimum order in the perturbative expansion, with associated genus g_0, gives,

$$\frac{4\Delta - 2d}{d-2} - s = 2g_0 - 2. \tag{8.54}$$

For $d = 6$

$$s = \Delta - 2g_0 - 1. \tag{8.55}$$

We then find

$$\begin{aligned}g_d^{s-2+2g_0}\Phi_p =& \sum_{k=1}^{4} V_n^{\frac{s}{2}} g_d^{2g_0-2} E_k P_k\left(\underline{\lambda}_1,\underline{\phi}\right) + \sum_{k=5}^{8} V_n^{-\frac{s}{2}+2} g_d^{2g_0-2} E_k P_k\left(\underline{\lambda}_3,\underline{\phi}\right) \\ &+ \sum_{k=9}^{12} V_n^{\frac{s}{2}-2} g_d^{2s-10+2g_0} E_k P_k\left(\underline{\lambda}_3,\underline{\phi}\right) + \sum_{k=13}^{16} V_n^{-\frac{s}{2}+4} g_d^{2s-10+2g_0} E_k P_k\left(\underline{\lambda}_1,\underline{\phi}\right).\end{aligned} \tag{8.56}$$

Comparing the power of g_d in terms 9 to 12 and 13 to 16 with g_d^{2g-2} one finds

$$2g - 2 = 2s - 10 + 2g_0. \tag{8.57}$$

After substituting for s we find the genus g as a function of the number of inverse metrics Δ and the genus g_0 at which the lowest order term in the automorphic form first appears

$$\begin{aligned}g &= s - 8 + g_0 \\ &= \Delta - 9 - g_0.\end{aligned} \tag{8.58}$$

So we see that the perturbative part of the automorphic form constructed from the **16** of $SO(5,5)$ with highest weight $\vec{\Lambda}^1$ is potentially valid for any higher derivative term in the type IIB effective action in $d = 10 - n$ dimensions satisfying $\Delta - 9 - g_0$ is a non-negative integer.

8.5.3 24 of $SL(5)$

The unconstrained Eisenstein-like automorphic form constructed from the representation of $SL(5)$ with highest weight $\vec{\Lambda}^{n-1} + \vec{\Lambda}^{n+1}$ is given by

$$
\begin{aligned}
\Phi_p = & \sum_{k=1}^{3} g_d^{-2s} V_n^{\frac{1}{3}s} E_k P_k\left(\underline{\lambda}_2, \phi\right) + \sum_{k=4}^{6} V_n^{\frac{4}{3}s-\frac{3}{2}} g_d^{-3} E_k P_k\left(\underline{\lambda}_2, \phi\right) \\
& + V_n^{-s+\frac{11}{2}} g_d^{-2s+3} E_7 + \sum_{k=8}^{10} V_n^2 g_d^{-4} E_k P_k\left(\left(\underline{\lambda}_1 + \underline{\lambda}_2, \phi\right)\right)^+ \\
& + V_n^2 g_d^{-4} E_{11} + V_n^2 g_d^{-4} E_{12} \\
& + V_n^2 g_d^{-4} E_{13} + V_n^2 g_d^{-4} E_{14} \\
& + \sum_{k=15}^{17} V_n^2 g_d^{-4} E_k P_k\left(\left(\underline{\lambda}_1 + \underline{\lambda}_2, \phi\right)\right)^- \\
& + V_n^{s-\frac{13}{2}} g_d^{2s-21} E_{18} \\
& + \sum_{k=19}^{21} V_n^{-\frac{4}{3}s+\frac{29}{2}} g_d^{-3} E_k P_k\left(\underline{\lambda}_1, \phi\right) \\
& + \sum_{k=22}^{24} V_n^{-\frac{1}{3}s+4} g_d^{2s-24} E_k P_k\left(\underline{\lambda}_1, \phi\right).
\end{aligned}
\tag{8.59}
$$

For any value of s we see that the perturbative part of the automorphic form contains odd and even powers of the d dimensional coupling g_d. It is therefore incompatible with a perturbative expansion in the d dimensional coupling and thus it appears that the unconstrained automorphic form constructed from the **24** of $SL(5)$ can not be the automorphic form of any higher derivative term in the effective action of type IIB string theory in $d = 7$ dimensions.

8.5.4 78 of E_6

The unconstrained Eisenstein-like automorphic form constructed from the representation of E_6 with highest weight $\vec{\Lambda}^{n-1}$ is given by

$$
\begin{aligned}
\Phi_p =& \sum_{k=1}^{5} V_n^{\frac{3}{5}s} g_d^{-2s} E_k P_k \left(\underline{\lambda}_4, \underline{\phi}\right) + \sum_{k=6}^{15} V_n^{-\frac{1}{5}s+2} g_d^{-2s} E_k P_k \left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ \sum_{k=16}^{25} V_n^{\frac{4}{5}s-\frac{11}{2}} g_d^{-15} E_k P_k \left(\underline{\lambda}_2, \underline{\phi}\right) + V_n^{-s+17} g_d^{-2s+10} E_{26} \\
&+ \sum_{k=27}^{36} V_n^{4} g_d^{-16} E_k P_k \left(\left(\underline{\lambda}^1 + \underline{\lambda}^4\right)^+, \underline{\phi}\right) \\
&+ V_n^4 g_d^{-16} E_{37} + V_n^4 g_d^{-16} E_{38} + V_n^4 g_d^{-16} E_{39} + V_n^4 g_d^{-16} E_{40} \\
&+ V_n^4 g_d^{-16} E_{41} + V_n^4 g_d^{-16} E_{42} + \sum_{k=43}^{52} V_n^{4} g_d^{-16} E_k P_k \left(\left(\underline{\lambda}^1 + \underline{\lambda}^4\right)^-, \underline{\phi}\right) \\
&+ V_n^{s+31} g_d^{2s-68} E_{53} + \sum_{k=54}^{63} V_n^{-\frac{4}{5}s+\frac{257}{10}} g_d^{-15} E_k P_k \left(\underline{\lambda}^3, \underline{\phi}\right) \\
&+ \sum_{k=64}^{73} V_n^{\frac{1}{5}s-\frac{29}{5}} g_d^{2s-78} E_k P_k \left(\underline{\lambda}^3, \underline{\phi}\right) \\
&+ \sum_{k=74}^{78} V_n^{-\frac{3}{5}s+\frac{117}{5}} g_d^{2s-78} E_k P_k \left(\underline{\lambda}^3, \underline{\phi}\right).
\end{aligned}
\tag{8.60}
$$

Demanding that the first term appears at some minimum order in the perturbative expansion, with associated genus g_0, gives,

$$
\frac{4\Delta - 2d}{d-2} - \frac{5}{3}s = 2g_0 - . \tag{8.61}
$$

For $d = 5$

$$
s = \frac{4}{5}\left(\Delta - 1\right) - \frac{6}{5}g_0. \tag{8.62}
$$

Thus,

$$
\begin{aligned}
g_d^{\frac{5}{3}s-2+2g_0}\Phi_p =& \sum_{k=1}^{5} V_n^{-\frac{s}{2}} g_d^{2g_0-2} E_k P_k\left(\underline{\lambda}_4, \underline{\phi}\right) + \sum_{k=6}^{15} V_n^{-\frac{13}{10}s+1} g_d^{2g_0-2} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ \sum_{k=16}^{25} V_n^{-\frac{3}{10}s-\frac{1}{2}} g_d^{2s-17+2g_0} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) \\
&+ \sum_{k=26}^{49} V_n^{-\frac{11}{10}s+\frac{9}{2}} g_d^{2s-17+2g_0} E_k P_k\left(\underline{\lambda}^1 + \underline{\lambda}^4, \underline{\phi}\right) \\
&+ V_n^{-\frac{21}{10}s+29} g_d^{32+2g_0} E_{50} + V_n^{-\frac{11}{10}s+5} g_d^{2s-16+2g_0} E_{51} \\
&+ V_n^{-\frac{11}{10}s+\frac{9}{2}} g_d^{2s-17+2g_0} E_{52} + V_n^{-\frac{1}{10}s-22} g_d^{4s-\frac{87}{2}+2g_0} E_{53} \\
&+ \sum_{k=54}^{63} V_n^{-\frac{19}{10}s+\frac{257}{10}} g_d^{2s-17+2g_0} E_k P_k\left(\underline{\lambda}^3, \underline{\phi}\right) \\
&+ \sum_{k=64}^{73} V_n^{-\frac{9}{10}s-\frac{33}{10}} g_d^{4s-75+2g_0} E_k P_k\left(\underline{\lambda}^3, \underline{\phi}\right) \\
&+ \sum_{k=74}^{78} V_n^{-\frac{14}{10}s+\frac{117}{5}} g_d^{4s-80+2g_0} E_k P_k\left(\underline{\lambda}^3, \underline{\phi}\right).
\end{aligned}
\tag{8.63}
$$

Comparing the power of g_d in terms 16 to 25 with g_d^{2g-2} we find

$$
2g-2 = 2s-17+2g_0. \tag{8.64}
$$

After substituting for s we find the genus g as a function of the number of inverse metrics Δ and the genus g_0 at which the lowest order term in the automorphic form first appears

$$
\begin{aligned}
g =& s - \frac{15}{2} + g_0 \\
=& \frac{8(\Delta-1) - 75 - 2g_0}{10}.
\end{aligned}
\tag{8.65}
$$

Examining the right hand side of the above equation, we see that the numerator is odd while the denominator is even. Therefore, g is not an integer for any number of inverse metrics Δ, and, as a consequence, Φ_p is not compatible with a perturbative expansion in g_d. Alternatively, one may observe that Φ_p contains even and odd powers of the d dimensional coupling for all s and g_0 and therefore the unconstrained automorphic form Φ constructed from the **78** of E_6 appears not to be a valid automorphic form for any higher derivative term in the effective action of type IIB string theory in $d = 5$ dimensions.

8.5.5 **45** of $SO(5,5)$

The unconstrained Eisenstein-like automorphic form constructed from the representation of $SO(5,5)$ with highest weight $\vec{\Lambda}^n$ is given by

$$\begin{aligned}\Phi_p =& V_n^s g_d^{-2s} E_1 + \sum_{k=2}^{7} V_n^{\frac{1}{2}} g_d^{-2s} E_k P_k \left(\underline{\lambda}^2, \underline{\phi}\right)\\
&+ \sum_{k=8}^{13} V_n^{s-3} g_d^{-7} E_k P_k \left(\underline{\lambda}_2, \underline{\phi}\right)\\
&+ V_n^{s-3} g_d^{-2s+6} E_{14} + \sum_{k=15}^{20} V_n^{-\frac{1}{2}s+\frac{7}{2}} g_d^{s-7} E_k P_k \left(\underline{\lambda}^1 + \underline{\lambda}^3, \underline{\phi}\right)\\
&+ \sum_{k=21}^{25} V_n^3 g_d^{-8} E_k\\
&+ \sum_{k=26}^{31} V_n^3 g_d^{-8} E_k P_k \left(\left(\underline{\lambda}^1 + \underline{\lambda}^3\right)^-, \underline{\phi}\right)\\
&+ V_n^{s+\frac{39}{2}} g_d^{2s-39} E_{32}\\
&+ \sum_{k=33}^{38} V_n^{-s+\frac{39}{2}} g_d^{-7} E_k P_k \left(\underline{\lambda}^2, \underline{\phi}\right)\\
&+ \sum_{k=39}^{44} V_n^{\frac{1}{2}} g_d^{2s-10} E_k P_k \left(\underline{\lambda}^2, \underline{\phi}\right)\\
&+ V_n^{-s+\frac{45}{2}} g_d^{2s-45} E_{45}.\end{aligned} \tag{8.66}$$

We find that the perturbative part of the unconstrained automorphic form constructed from the **45** of $SL(5)$ for any value of s contains odd and even powers of the d dimensional coupling g_d and is therefore incompatible with a perturbative expansion in the d dimensional coupling. It follows that the unconstrained automorphic form constructed from the **45** of $SL(5)$ may not appear as the automorphic form of any higher derivative term in the effective action of type IIB string theory in $d = 6$ dimensions.

8.5.6 248 of E_8

The unconstrained Eisenstein-like automorphic form constructed from the representation of E_8 with highest weight $\vec{\Lambda}^1$ is given by

$$
\begin{aligned}
\Phi_p =& \sum_{k=1}^{7} V_n^{\frac{2s}{7}} g_d^{-4s} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=8}^{14} V_n^{-\frac{2}{7}s+2} g_d^{-4s} E_k P_k\left(\underline{\lambda}^6, \underline{\phi}\right) \\
&+ \sum_{k=15}^{21} V_n^{\frac{5}{7}s-5} g_d^{-2s-14} E_k P_k\left(\underline{\lambda}^6, \underline{\phi}\right) \\
&+ \sum_{k=22}^{56} V_n^{\frac{1}{7}s+1} g_d^{-2s-14} E_k P_k\left(\underline{\lambda}^4, \underline{\phi}\right) \\
&+ \sum_{k=57}^{77} V_n^{-\frac{3}{7}s+17} g_d^{-2s-14} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ \sum_{k=78}^{98} V_n^{\frac{4}{7}s+\frac{43}{2}} g_d^{-91} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ V_n^{-s+\frac{111}{2}} g_d^{-2s+7} E_{99} \\
&+ \sum_{k=100}^{120} V_n^{6} g_d^{-92} E_k P_k\left(\left(\underline{\lambda}^1 + \underline{\lambda}^6\right)^+, \underline{\phi}\right) \\
&+ \sum_{k=121}^{128} V_n^{6} g_d^{-92} E_k \\
&+ \sum_{k=129}^{149} V_n^{6} g_d^{-92} E_k P_k\left(\left(\underline{\lambda}^1 + \underline{\lambda}^6\right)^-, \underline{\phi}\right) \\
&+ V_n^{s-\frac{137}{2}} g_d^{2s-241} E_{150} \\
&+ \sum_{k=151}^{171} V_n^{-\frac{2}{7}s+\frac{691}{14}} g_d^{-91} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right) \\
&+ \sum_{k=172}^{192} V_n^{\frac{3}{7}s-\frac{253}{7}} g_d^{2s-262} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right) \\
&+ \sum_{k=193}^{227} V_n^{-\frac{1}{7}s+\frac{131}{7}} g_d^{2s-262} E_k P_k\left(\underline{\lambda}^3, \underline{\phi}\right) \\
&+ \sum_{k=228}^{234} V_n^{-\frac{5}{7}s+\frac{585}{7}} g_d^{2s-262} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) \\
&+ \sum_{k=235}^{241} V_n^{\frac{2}{7}s-\frac{234}{7}} g_d^{4s-496} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) \\
&+ \sum_{k=242}^{248} V_n^{-\frac{2}{7}s+\frac{248}{7}} g_d^{4s-496} E_k P_k\left(\underline{\lambda}^6, \underline{\phi}\right) .
\end{aligned}
\tag{8.67}
$$

The evaluation of the perturbative part of the unconstrained automorphic form constructed from the **248** of E_8 demonstrates that any higher derivative term carrying this automorphic form possesses odd and even powers of the effective coupling g_d for any value of s and thus appears to be incompatible with a perturbative expansion in the effective coupling g_d. This suggests that the unconstrained automorphic form constructed from the **248** of E_8 could not appear as an automorphic form for any higher derivative term in the effective action of type IIB string theory in three dimensions.

8.5.7 56 of E_7

The unconstrained Eisenstein-like automorphic form constructed from the representation of E_7 with highest weight $\vec{\Lambda}^1$ is given by

$$\begin{aligned}\Phi_p =& \sum_{k=1}^{6} V_n^{\frac{s}{3}} g_d^{-2s} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=7}^{12} V_n^{-\frac{1}{3}s+2} g_d^{-2s} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right) \\ &+ \sum_{k=13}^{18} V_n^{\frac{2}{3}s-4} g_d^{-12} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) + \sum_{k=19}^{38} V_n^{2} g_d^{-12} E_k P_k\left(\underline{\lambda}^3, \underline{\phi}\right) \\ &+ \sum_{k=39}^{44} V_n^{-\frac{2}{3}s+\frac{44}{3}} g_d^{-12} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) + \sum_{k=45}^{50} V_n^{\frac{1}{3}s-\frac{22}{3}} g_d^{2s-56} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) \\ &+ \sum_{k=51}^{56} V_n^{-\frac{1}{3}s+\frac{28}{3}} g_d^{2s-56} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right).\end{aligned} \tag{8.68}$$

Demanding that the first term appears at some minimum order in the perturbative expansion, with associated genus g_0, gives,

$$\frac{4\Delta - 2d}{d-2} - 2s = 2g_0 - 2. \tag{8.69}$$

For $d = 4$

$$s = \Delta - 1 - g_0. \tag{8.70}$$

Therefore we have,

$$\begin{aligned}g_d^{2s-2+2g_0}\Phi_p =& \sum_{k=1}^{6} V_n^{\frac{s}{3}} g_d^{-2+2g_0} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=7}^{12} V_n^{-\frac{1}{3}s+2} g_d^{-2+2g_0} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right) \\ &+ \sum_{k=13}^{18} V_n^{\frac{2}{3}s-4} g_d^{2s-14+2g_0} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) + \sum_{k=19}^{38} V_n^{2} g_d^{2s-14+2g_0} E_k P_k\left(\underline{\lambda}^3, \underline{\phi}\right) \\ &+ \sum_{k=39}^{44} V_n^{-\frac{2}{3}s+\frac{44}{3}} g_d^{2s-14+2g_0} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) + \sum_{k=45}^{50} V_n^{\frac{1}{3}s-\frac{22}{3}} g_d^{4s-58+2g_0} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) \\ &+ \sum_{k=51}^{56} V_n^{-\frac{1}{3}s+\frac{28}{3}} g_d^{4s-58+2g_0} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right).\end{aligned} \tag{8.71}$$

Comparing the power of g_d in terms 13 to 43 with g_d^{2g-2} we find

$$2g - 2 = 2s - 14 + 2g_0 \tag{8.72}$$

After substituting for s we find the genus g_1, at which terms 13 to 43 receive perturbative contributions, as a function of the number of inverse metrics Δ and the genus g_0 at which the lowest order term in the automorphic form first appears

$$\begin{aligned} g_1 =& \Delta - 1 - g_0 - 6 + g_0 \\ =& \Delta - 7. \end{aligned} \tag{8.73}$$

Performing similar analysis for terms 44 to 56 by comparing the power of g_d in terms 44 to 56 with $g_d^{2g_2-2}$ we find

$$2g_2 - 2 = 4s - 58 + 2g_0. \tag{8.74}$$

The genus g_2 at which terms 44 to 56 receive perturbative contributions as a function of the number of inverse metrics Δ and the genus g_0 associated with the perturbative contributions of the lowest order term in the automorphic form is then

$$\begin{aligned} g_2 =& 2s - 28 + g_0 \\ =& 2\Delta - 30 - g_0. \end{aligned} \tag{8.75}$$

Therefore the automorphic form constructed from the **56** of E_7 with highest weight $\vec{\Lambda}^2$ could appear as the coefficient function for any higher derivative term in the type IIB effective action in $d = 10-n$ dimensions if $2\Delta - 30 - g_0$ is a non-negative integer. Note that any higher derivative term satisfying this condition automatically satisfies the weaker condition on the genus g_1 associated with terms 13 to 43.

8.6 Conclusion

From the evaluation of the perturbative parts of the automorphic forms in section 8.5 it appears that the automorphic forms constructed from the **10** of $SL(5)$, **16** of $SO(5,5)$ and the **56** of E_7 are compatible with a perturbative expansion in the effective coupling g_d in d dimensions for particular higher derivative terms. Therefore these automorphic forms seem to be potentially acceptable coefficient functions for particular sets of higher derivative terms.

The automorphic forms constructed from the **24** of $SL(5)$, **45** of $SO(5,5)$, **78** of E_6 and **248** of E_8 do not appear to be compatible with a perturbative expansion in the effective coupling g_d in $d = 10 - n$ dimensions for any higher derivative term. One would be tempted to conclude that the automorphic forms constructed from these representations do not appear as the coefficient functions of any higher derivative term in the effective action of type IIB string theory in $d = 10 - n$ dimensions. However, in evaluating the perturbative part of the automorphic form we have somewhat arbitrarily chosen an ordering of the simple roots of the E_{n+1} algebra, such that $\vec{\alpha}_n > \vec{\alpha}_{n+1} > \vec{\alpha}_j$ for $j = 1, 2, ..., n-1$. Although the complete automorphic form is independent of any choice of ordering of the root space, once the automorphic form is split into a perturbative part and a non-perturbative part by writing $\Phi = \Phi_p + \Phi_{np}$, one finds that the perturbative and non-perturbative parts of the automorphic form are, in general, dependent on the chosen ordering of the E_{n+1} root space. This ambiguity in the ordering of the root space is not an issue for automorphic forms constructed from a representation of an E_{n+1} algebra with a root string that does not split, in the sense that for all weights $\vec{\mu}$ in the root string of a highest weight representation of E_{n+1} there is at most one simple root $\vec{\alpha}_i$ satisfying $(\vec{\alpha}_i, \vec{\mu}) = 1$, where $i = 1, 2, ..., n+1$, and no simple roots satisfying $(\vec{\alpha}_i, \vec{\mu}) > 1$. It follows that since the automorphic form $\Phi_{SL(5)}$ that appears as the coefficient function for the R^4 term in the type IIB effective action in $d = 7$ dimensions is constructed from the **5** of $SL(5)$, which possesses a root string that does not split, one finds that the perturbative part of the automorphic form as evaluated in reference [40] is defined unambiguously.

The coefficient function for the R^4 term in the type IIB effective action in $d = 6$ dimensions is constructed from the **10** of $SO(5,5)$, which does possess a root string that splits. However, the ambiguity in the ordering of the weights is lifted once a constraint on the $SO(5,5)$ lattice, as mentioned in section 8.2 and evaluated in reference [40], is implemented.

In chapter 9 we study the behaviour of automorphic forms in the limits of several physical parameters. In particular, one may think of the $d = 10-n$ dimensional weak coupling limit $g_d \to 0$, described in chapter 9, as an alternative definition of the perturbative part of the automorphic form that is unambiguously defined but unlike the definition outlined in this chapter retains some dependence on the axionic fields $\chi_{\vec{\alpha}}$ and possesses an explicit $SO(n,n)$ symmetry.

9 Limits

The higher derivative terms in the effective action of type IIA/B string theory and M-theory must match the known properties of the $d = 10 - n$ dimensional type IIA/B string theory and M-theory effective actions in the limits of various physical parameters. The automorphic forms that act as the coefficient functions of the higher derivative terms in the effective action of type IIA/B string theory and M-theory in $d = 10 - n$ dimensions are, in general, dependent on these parameters and thus one may gain insights into which automorphic forms are acceptable coefficient functions of higher derivative terms by investigating their behaviour in the limits of these physical parameters.

In this chapter we explore these limits in the context of the higher derivative terms of the effective action of type IIA/B string theory and M-theory through the E_{11} formulation of type IIA/B and eleven dimensional supergravity, in taking this approach we are able to attach a group theoretical meaning to each limit. We then derive a general formula for the unconstrained Eisenstein-like automorphic forms, described in chapter eight, in various limits of the physical parameters and explicitly evaluate the limits of the unconstrained Eisenstein-like automorphic forms constructed from the $\mathbf{5}$ of $SL(5)$ and the perturbative limits of the $\mathbf{133}$ of E_7 and the $\mathbf{248}$ of E_8. The unconstrained Eisenstein-like automorphic form constructed from the $\mathbf{133}$ of E_7 is an example of an automorphic form that, in these limits, violates known properties of type IIA/B string theory dimensionally reduced to $d = 4$ dimensions and so can not appear as the coefficient function of any higher derivative term in the effective action of type IIA/B string theory and M-theory in $d = 4$ dimensions.

The unconstrained Eisenstein-like automorphic form constructed from the $\mathbf{5}$ of $SL(5)$ is an example of an automorphic form that does appear as the coefficient function of known higher derivative terms in the effective action of type IIA/B string theory and M-theory in $d = 7$ dimensions. We consider the conditions under which the unconstrained Eisenstein-like automorphic form constructed from the $\mathbf{5}$ of $SL(5)$ could appear as a coefficient function in the $d = 7$ type IIA/B string theory and M-theory effective actions for higher order terms beyond those already known.

For an example of how the limits of physical parameters may constrain the higher derivative terms consider the R^4 term in the $d = 10$ type IIB string theory effective action. The coefficient function of the R^4 term in the ten dimensional type IIB effective action is an $SL(2)$ Eisenstein series with $s = \frac{3}{2}$ that may be constructed from an $SL(2)/SO(2)$ group element parameterised by the type IIB dilaton ϕ and axion χ. The type IIB string coupling g_s is defined by $g_s = e^{\phi}$ and therefore the $SL(2)$ Eisenstein series of the R^4 term is a function of the type IIB string coupling g_s. In the weak coupling limit $g_s \to 0$, contributions to the effective action of type IIB string theory in ten dimensions from non-perturbative effects are suppressed and the remaining terms in the effective action are of a perturbative origin and thus, in string frame, must be multiplied by a factor of $e^{(-2+2k)\phi}$, where k is a non-negative integer and corresponds to the genus of the perturbative contribution. In the weak coupling limit the R^4 term that possesses an $SL(2)$ Eisenstein series with $s = \frac{3}{2}$ coefficient function reduces to a sum of two terms at tree level and one loop with type IIB string coupling dependence g_s^{-2} and g_s^0, in string frame. Since the weak coupling limit of the R^4 term carrying a coefficient function given by the $SL(2)$ Eisenstein series with $s = \frac{3}{2}$ reproduces the known result that the R^4 term picks up perturbative contributions at tree level and one loop only, we see that in principle the $SL(2)$ Eisenstein series with $s = \frac{3}{2}$ could be the coefficient function of the R^4 term. However, if we evaluated the weak coupling limit $g_s \to 0$ of the R^4 term carrying a coefficient function given by a different $SL(2)$ automorphic form we would not be guaranteed either that the resulting perturbative contributions would agree with the known results or even a set of terms that could be identified with a perturbative expansion in g_s, i.e. terms that are multiplied by a factor of $e^{(-2+2k)\phi}$.

In $d = 10 - n$ dimensions the higher derivative terms in the effective action of type IIA/B string theory and M-theory possess coefficient functions that transform as E_{n+1} automorphic forms that are dependent on the effective coupling g_d and the moduli of the n torus. In addition to considering the d dimensional perturbative limit $g_d \to \infty$, one may also consider various decompactification limits of the moduli of the n torus. The various decompactification limits also place demands on the automorphic forms of the higher derivative terms.

References [45–47] investigated the behaviour of the $d = 10-n$ dimensional R^4, $\partial^4 R^4$ and $\partial^6 R^4$ terms with their associated coefficient function in several of these limits. The coefficient functions of the $d = 10 - n$ dimensional R^4, $\partial^4 R^4$ and $\partial^6 R^4$ terms are protected by supersymmetry, which constrains them to satisfy Poisson equations on the E_{n+1}/H moduli space, and in each case the proposed coefficient functions were shown to agree with known type IIB string theory and M-theory results in the appropriate limits.

Higher derivative terms in the effective action of type IIA/B string theory and M-theory beyond the $\partial^6 R^4$ term are not known to be protected by supersymmetry and therefore any possible Poisson equations constraining the coefficient functions of these terms are unknown in general, although reference [33] puts forward arguments for the Poisson equations satisfied by the $\partial^{10} R^4$, $\partial^{12} R^4$ and $\partial^{14} R^4$ terms in the ten dimensional type IIB string theory effective action. However, even without knowing the form of the Poisson equations satisfied by the coefficient functions of these higher order terms, one may still gain an insight into their possible structure by studying the behaviour of the higher derivative terms in the limits of various physical parameters.

The E_{11} formulation of type IIA/B supergravity and eleven dimensional supergravity provides a common framework for discussion of the limits of the higher derivative terms in effective actions of type IIA/B string theory and M-theory. In particular, the E_{n+1} Cartan subalgebra fields are in one to one correspondence with the physical fields, i.e. the type IIA/B dilaton and the n torus moduli, that determine the physical parameters. As a result, the physical parameters may be formulated in terms of the fields associated with the Cartan subalgebra of E_{n+1} that naturally arises as a subalgebra of E_{11}. Moreover, the E_{11} formulation of type IIA/B supergravity and eleven dimensional supergravity differs only by the chosen decomposition of the E_{11} subalgebra and in $d < 10$ dimensions the type IIA/B and eleven dimensional supergravity theories are considered equivalent and are all obtained by deleting node d in the E_{11} Dynkin diagram to reproduce an $SL(d)$ gravity line. A consequence of the type IIA/B and eleven dimensional supergravity theories being equivalent once compactified on a torus to $d < 10$ dimensions is that there is no preferred theory for which to consider the $d = 10 - n$ dimensional effective action. So, as we will see, the various limits in the physical parameters may be given a group theoretical meaning that is independent of the physical fields, i.e. the sets of type IIA/B or M-theory fields, one may wish to use in describing the d dimensional theory.

We will begin by reviewing the physical parameters in the effective actions of type IIA/B string theory and M-theory and their expressions in terms of the physical fields of these theories. We then derive the relationship between the physical fields of type IIA/B supergravity and eleven dimensional supergravity in d dimensions and the fields that parameterise the group elements in the E_{11} formulations of type IIA/B supergravity and eleven dimensional supergravity, in this way we can relate the physical parameters in the effective actions of type IIA/B string theory and M-theory to the E_{11} fields associated with the Cartan subalgebra. The physical parameters are found to be associated with specific fields that parameterise the E_{n+1} subgroup contained in E_{11}. Moreover, the fields used to parameterise the E_{n+1} subgroup are defined in Chevalley basis, where they are naturally associated with nodes of the E_{n+1} Dynkin diagram. As a result, the limit in each parameter can be given a group theoretical meaning.

We then consider the general behaviour of an arbitrary higher derivative term in the effective action of type IIA/B string theory in each of these limits. We also derive an explicit general form for the unconstrained Eisenstein-like automorphic forms constructed from highest weight representations of E_{n+1} in the various limits of the physical parameters. To demonstrate how one may make use of these limits to investigate the role of an automorphic form in the effective action of type IIA/B string and M-theory we examine the unconstrained Eisenstein-like automorphic forms constructed from the **5** of $SL(5)$, the **133** of E_7 and the **248** of E_8. .

9.1 Parameters

The parameters of interest to us in taking various limits of type IIA and IIB string theory and M-theory are the d dimensional Planck length l_d, the string length l_s, the IIA and IIB string coupling, $g_{s(A)}$ and $g_{s(B)}$, the effective coupling g_d in $d = 10 - n$ dimensions, the radii r_i, $i = d + 1, ..., D$, of the n torus and volume of the torus upon which type IIA string theory, type IIB string theory and M-theory are compactified on, $V_{n(A)}$, $V_{n(B)}$ and V_m respectively. The Planck length in eleven dimensions and the the IIA and IIB Planck length in ten dimensions are denoted by l_{11}, $l_{10(A)}$ and $l_{10(B)}$ respectively. The type IIA/B supergravity theories have two parameters, the Newtonian coupling constant κ_{10} and $e^{<\phi>}$, where $<\phi>$ is the expectation value of the type IIA/B dilaton ϕ. The d dimensional Newtonian coupling constant κ_d appears in the action, multiplying the d dimensional Einstein-Hilbert term in Einstein frame, in the form

$$\frac{1}{2\kappa_d^2}\int d^d x\sqrt{-g}R. \tag{9.1}$$

By dimensional analysis the d dimensional Planck length l_d is defined by

$$l_d^{d-2} = 2\kappa_d^2. \tag{9.2}$$

Type IIA/B supergravity are the low-energy effective theories derived from type IIA/B string theory, respectively. The type IIA/B string theories also have two parameters, the string length l_s and the string coupling constant $g_s = e^{<\phi>}$, where the string length may be defined in terms of α' as $l_s = \sqrt{\alpha'}$. The relationship between the type IIA and IIB supergravity coupling κ_{10} and the string length l_s is found by comparing the supergravity action and the effective action derived from type IIA and IIB string theory, this yields,

$$2\kappa_{10}^2 = (2\pi)^7 l_s^8 g_s^2, \tag{9.3}$$

and so one finds

$$l_{10}^8 = (2\pi)^7 l_s^8 g_{(s)A/B}^2. \tag{9.4}$$

We will dimensionally reduce type IIA/B string theory and M-theory by taking our standard torus ansatz of

$$d\hat{s}^2 = e^{2\alpha\rho}ds^2 + e^{2\beta\rho}G_{ij}\left(dx^i + A^i_\mu dx^\mu\right)\left(dx^j + A^j_\mu dx^\mu\right), \tag{9.5}$$

where $det(G) = 1$ and the constants α and β are defined in equations (3.12) and (3.13), respectively. Compactifying a D dimensional theory to $d = D - n$ dimensions, via this ansatz, gives the d dimensional theory in Einstein frame. The coordinate and parameterisation invariant length of a torus cycle in the i direction is

$$l_d\int \hat{e}_i{}^i dx^i = r_i, \tag{9.6}$$

where l_d is the d dimensional Planck length and r_i is the radius in the i direction. The vielbein of the internal metric $\hat{e}_i{}^i$ is taken to be independent of the compactified coordinates, therefore the integration is trivial and one finds

$$\hat{e}_i{}^i = \frac{r_i}{l_d}. \tag{9.7}$$

The volume of the n torus compactifying a D dimensional theory to $d = D - n$ dimensions may then be written $V_n = \frac{r_D r_{D-1} \ldots r_{d+1}}{(l_D)^n} = e^{n\beta\rho}$, in Einstein frame. Note that in $d = 10 - n$ dimensional string frame the volume of the n torus is given by $V_{n(s)} = e^{n\beta\rho + \frac{n}{4}\phi}$. The effective coupling in d dimensions is defined by

$$\begin{aligned} g_d &= \alpha'^{\frac{n}{4}} g_s V_{n(s)}^{-\frac{1}{2}} \\ &= e^{\frac{8-n}{8}\phi - \frac{n\beta\rho}{2}}, \end{aligned} \tag{9.8}$$

where, in this expression, V_n is the n torus volume in string frame.

The $D = 11$ metric used to compactify eleven-dimensional supergravity to IIA supergravity is a special case of the above ansatz and is given by

$$ds^2 = e^{-\frac{2}{3}\phi} g_{\mu\nu} dx^\mu dx^\nu + e^{\frac{4}{3}\phi} \left(dx^{11} + A_\mu dx^\mu\right)^2. \tag{9.9}$$

From this metric we may identify $e^{\frac{2}{3}\phi}$ as the vielbein on the circle $\hat{e}_{11}^{\ 11}$ and so one finds

$$r_{11} = g_{s(A)}^{\frac{2}{3}} l_{11}. \tag{9.10}$$

Compactifying on a circle of radius r we find that the relationship between the d dimensional Newtonian coupling constant and the $d+1$ dimensional Newtonian coupling constant is

$$(\kappa_{d+1})^2 = 2\pi r (\kappa_d)^2. \tag{9.11}$$

It then follows that upon compactification of the eleventh dimension, we have

$$2\kappa_{11}^2 = 4\pi r_{11} \kappa_{10}^2 = 4\pi r_{11} l_s^8 g_{s(A)}^2, \tag{9.12}$$

where κ_{11} is the $d = 11$ supergravity coupling constant. Using our expression for the eleven dimensional Planck length l_{11} in terms of the IIA string coupling $g_{s(A)}$ and the radius of the eleventh dimension r_{11}, we find, from the above expression,

$$r_{11} = g_s l_s. \tag{9.13}$$

One may rewrite equation (9.11) in terms of the Planck length rather than the Newtonian coupling constant, this leads to

$$(l_{d+1})^{d-1} = 2\pi r (l_d)^{d-2}. \tag{9.14}$$

We may write the $d = 10 - n$ dimensional Planck length in terms of the volume of an n-torus and the ten dimensional Planck length, by iterating equation (9.14) n times and dividing by $(l_{10})^n$, this gives $l_d^{d-2} = \frac{l_{10}^{d-2}}{V_n}$. Our expressions for the volume V_n, coupling constant g_d (9.8) and the ten dimensional Planck length l_{10} (9.4) then allow us to write

$$l_d^{d-2} = g_d^2 l_s^{d-2}. \tag{9.15}$$

Similarly, we may iterate equation (9.14) $n+1$ times to the find the d dimensional Planck length l_d as a function of the volume of the torus used to dimensionally reduce M-theory and the eleven dimensional Planck length l_{11}, this yields

$$l_d = l_{11} \hat{V}_{n+1}^{-\frac{1}{8-n}}. \tag{9.16}$$

In summary, we have that the Planck length in ten and eleven dimensions are related to the string length and IIA/IIB coupling by

$$l_{11} = g_{s(A)}^{\frac{1}{3}} l_s, \tag{9.17}$$

$$l_{10(A)} = g_{s(A)}^{\frac{1}{4}} l_s, \tag{9.18}$$

$$l_{10(B)} = g_{s(B)}^{\frac{1}{4}} l_s, \tag{9.19}$$

$$r_{11} = g_{s(A)} l_s, \tag{9.20}$$

While the d dimensional Planck length l_d is related to the d dimensional coupling g_d, the type IIA torus volume $V_{n(A)}$, type IIB torus volume $V_{n(B)}$, M-theory torus volume V_m, and the radius r_{d+1} on which a $d+1$ dimensional theory is compactified on by

$$l_d = r_{d+1}^{-\frac{1}{d-2}} \left(l_{d+1}\right)^{\frac{d-1}{d-2}}, \tag{9.21}$$

$$l_d = l_{10} V_{n(A)}^{-\frac{1}{8-n}}, \tag{9.22}$$

$$l_d = l_{10} V_{n(B)}^{-\frac{1}{8-n}}, \tag{9.23}$$

$$l_d = l_{11} V_m^{-\frac{1}{8-n}}, \tag{9.24}$$

$$l_d = g_d^{\frac{2}{8-n}} l_s. \tag{9.25}$$

9.2 E_{11}, E_{n+1} and the Parameters of Dimensionally Reduced Maximal Supergravity

In this section we will use the E_{11} formulation of type IIA/B and eleven dimensional supergravity to derive the relationship between the fields that parameterise the E_{n+1} group element, used to construct the E_{n+1} automorphic forms in $d = 10 - n$ dimensions, and the physical parameters discussed in section (9.1). In particular, we will see that the fields $\dot{\varphi}_i$, $i = d{+}1, ..., 11$, parameterising the Cartan subalgebra part of the E_{n+1} group element, in Chevalley basis, are directly related to the physical parameters of type IIA/B and eleven dimensional supergravity. Since we work with the E_{n+1} fields in Chevalley basis we find the physical parameters are associated with various nodes of the E_{n+1} Dynkin diagram.

The eleven dimensional, IIA and IIB supergravity theories, as well as the maximal type II supergravity theories in lower dimensions, can be formulated as non-linear realisations [66–71]. The non-linear realisations of the Kac-Moody algebra E_{11}, at low levels, leads to all of these theories. As such E_{11} encodes the fields of each of these theories and provides us with a way of relating the fields in the different theories to each other. In fact the fields of these theories are in one to one correspondence with the generators of the Borel subalgebra of E_{11} in the group decomposition, explained below, appropriate to each theory. It has been conjectured that non-linear realisations of the Kac-Moody algebra E_{11} are extensions of all these supergravity theories [65].

A Kac-Moody algebra is formulated in terms of its Chevalley generators, which include those in the Cartan subalgebra denoted by $H_a,\ a = 1, 2, \ldots 11$. As such, the E_{11} group element that occurs in the non-linear realisation is of the form $g_{E_{11}} = e^{\sum_a \phi_a H_a}$ provided we restrict our attention to the part that is in the Cartan subalgebra.

9.2.1 Type IIB in $d = 10$ Dimensions

The E_{11} formulation of type IIB supergravity is given by decomposing the E_{11} algebra in terms of the algebra that results from deleting the node labelled nine in the E_{11} Dynkin diagram in figure 17, namely the algebra $GL(10)$. Deleting node nine gives the gravity line in the type IIB theory

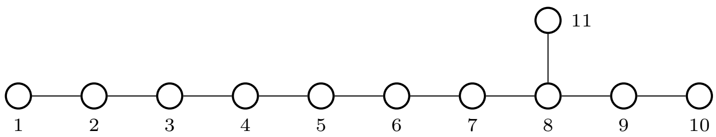

Figure 17: The E_{11} Dynkin diagram

that consists of nodes one to eight in addition to node 11 in the Dynkin diagram and an internal $SL(2)$ symmetry through node 10. The $GL(10)$ subalgebra is generated by $\hat{K}^a{}_b,\ a, b = 1, \ldots, 10$. Together with the generator $\hat{R}$ of the $SL(2)$ symmetry, the generators $\hat{K}^a{}_a$, $a = 1, ..., 10$ provide a basis for the Cartan subalgebra generators of E_{11}. The Cartan subalgebra generators in Chevalley basis for the IIB theory are

$$
\begin{aligned}
\hat{H}_a &= \hat{K}^a{}_a - \hat{K}^{a+1}{}_{a+1}, \quad a = 1, ..., 8, \\
\hat{H}_9 &= \hat{K}^9{}_9 + \hat{K}^{10}{}_{10} - \frac{1}{4}\sum_{a=1}^{10} \hat{K}^a{}_a + \hat{R}, \\
\hat{H}_{10} &= -2\hat{R}, \\
\hat{H}_{11} &= \hat{K}^9{}_9 - \hat{K}^{10}{}_{10}.
\end{aligned}
\tag{9.26}
$$

One may therefore write the E_{11} group element restricted to the Cartan subalgebra describing the type IIB theory $e^{\sum_a \hat{\phi}_a \hat{H}_a}$, where $\hat{\phi}$ are the fields associated with the Chevalley generators $\hat{H}_a$.

The E_{11} group element in the non-linear realisation for the type IIB theory is of the form

$$g = e^{x^a P_a} e^{h_a{}^b K^a{}_b} e^{\hat{R}\hat{\sigma}}, \tag{9.27}$$

where we have added the spacetime translation generators P_a and $\hat{\sigma}$ is the type IIB dilaton. The Cartan form for this subgroup is given by

$$g^{-1}dg = dx^\mu e_\mu{}^a P_a + (e^{-1}de)_a{}^b K^a{}_b, \tag{9.28}$$

where one finds that $e_\mu{}^a = (e^h)_a{}^b$ is the ten dimensional vielbein.

Restricting to the Cartan subalgebra we may set the different formulations of the E_{11} group element to be equal so that

$$e^{\sum_a \hat{\phi}_a \hat{H}_a} = e^{\sum_a h_a{}^a K^a{}_a} e^{\hat{\sigma}\hat{R}}. \tag{9.29}$$

Comparing the coefficients of the generator R we find

$$\hat{\phi}_{10} = -\frac{1}{2}\hat{\sigma}, \tag{9.30}$$

one may similarly compare the coefficients of the $GL(10)$ generators to find an expression for the rest of the E_{11} Chevalley fields in terms of the physical fields $\hat{h}^a{}_a$, $a = 1, ..., 10$.

9.2.2 Dimensionally Reduced type IIB

The type IIB E_{11} group element k_{IIB} in the dimensionally reduced theory is found by deleting node d of the Dynkin diagram. As in the eleven dimensional supergravity case, this splits the E_{11} algebra into a gravity sub-algebra, associated with the generators of $SL(d)$ restricted to the Cartan subalgebra $K^a{}_a$, $a = 1, ..., d$ and a remaining internal sub-algebra, consisting of all the generators lying in the Cartan sub-algebra that are not elements of the gravity subalgebra. The internal subalgebra generates the E_{n+1} symmetry group in $d = 10 - n$ dimensions. The generators associated with the gravity line are the $SL(d)$ generators, $\hat{K}^a{}_b$, $a, b = 1, ...d$, while the generators $\dot{K}^i{}_j = \hat{K}^i{}_j - \frac{1}{8-n}\delta^i_j \sum_{a=1}^d \hat{K}^a{}_a$, for $i, j = d+1, ..., 10$, and $\hat{R}$ form the E_{n+1} internal subalgebra. One may verify that the generators of the internal subalgebra satisfy $[P_a, \dot{K}^i{}_j] = 0$, for all $a = 1, ..., d$ and $i, j = d + 1, ..., 10$, where P_a are the d dimensional spacetime translation generators with commutation relations specified in equation (4.61). The group element k_{IIB} may then be written in the form

$$k_{IIB} = e^{\sum_{a=1}^d \hat{h}^a{}_a \hat{K}^a{}_a} e^{\sum_{i=d+1}^{11} \dot{\varphi}_i \hat{T}_i}, \tag{9.31}$$

where $\dot{\varphi}_a$ are the E_{n+1} Chevalley fields that parameterise the E_{n+1} Cartan subalgebra and

$$\begin{aligned}
\hat{T}_i &= \dot{K}^{i+1}{}_{i+1} - \dot{K}^{i+2}{}_{i+2} \quad for\ i = d+1, ..., 8, \\
\hat{T}_9 &= -\frac{1}{4}\left(\sum_{i=d+1}^{8} \dot{K}^i{}_i\right) + \frac{3}{4}\left(\dot{K}^9{}_9 + \dot{K}^{10}{}_{10}\right) + \hat{R}, \\
\hat{T}_{10} &= -2\hat{R}, \\
\hat{T}_{11} &= \dot{K}^9{}_9 - \dot{K}^{10}{}_{10}.
\end{aligned} \tag{9.32}$$

Substituting the expression $\dot{K}^i{}_j = \hat{K}^i{}_j - \frac{1}{8-n}\delta^i_j \sum_{a=1}^{d} \hat{K}^a{}_a$ into equation (9.32) one finds $\hat{H}_i = \hat{T}_i$ for $i = d+1, ..., 11$.

To find the E_{n+1} Chevalley fields parameterising k_{IIB} in the dimensionally reduced theory in terms of the physical fields that appear in $d = 10-n$ dimensions we compare the E_{11} group element k_{IIB} to the E_{11} group element g that corresponds to our chosen compactification ansatz (9.5).

The E_{11} group element that contains the type IIB physical fields in the dimensionally reduced theory after taking the metric compactification ansatz in equation (9.5) is given by

$$g = e^{x^\mu P_\mu} e^{\hat{h}^a{}_a \hat{K}^a{}_a + e_1 \hat{\rho} \sum_{a=1}^{d} \hat{K}^a{}_a} e^{\hat{h}^i{}_i \hat{K}^i{}_i + e_2 \hat{\rho} \sum_{i=d+1}^{10} \hat{K}^i{}_i} e^{\hat{R}\hat{\sigma}}. \tag{9.33}$$

Evaluating the Cartan form gives,

$$g^{-1}\left(dx^a P_a + dx^i P_i\right) g = \sum_{a=1}^{d}\left(e^{\hat{h}^a{}_a + \hat{e}_1 \hat{\rho}}\right) dx^a P_a + \sum_{i=d+1}^{10}\left(e^{\hat{h}^i{}_i + \hat{e}_2 \hat{\rho}}\right) dx^i P_i, \tag{9.34}$$

up to the same numerical for each $\hat{h}^i{}_i$ and neglecting terms involving derivatives of $\hat{h}^i{}_i$. The Cartan form $\mathcal{V}$ in d dimensions, restricted to the Cartan sub-algebra, may be expanded as

$$\mathcal{V} \equiv dx^\mu \hat{e}_\mu{}^a P_a + dx^i \hat{e}_i{}^j P_j. \tag{9.35}$$

From our compactification ansatz, with the one-form gauge fields A set to zero, we may identify $\hat{e}_\mu{}^a = e^{\hat{\alpha}\hat{\rho}} e_\mu{}^a$ and $\hat{e}_j{}^i = e^{\hat{\beta}\hat{\rho}} e_j{}^i$. Upon comparing the Cartan form gdg^{-1} and the expansion of the Cartan form $\mathcal{V}$, with the above identifications of the vielbeins, we find

$$\sum_{a=1}^{d}\left(e^{\hat{h}^a{}_a + \hat{e}_1 \hat{\rho}}\right) dx^a P_a + \sum_{i=d+1}^{10}\left(e^{\hat{h}^i{}_i + \hat{e}_2 \hat{\rho}}\right) dx^i P_i = e^{\alpha\rho} e_\mu{}^a dx^\mu P_a + e^{\beta\rho} e_j{}^i dx^j P_i, \tag{9.36}$$

from which, it is apparent $\hat{e}_1 = \alpha$, $\hat{e}_2 = \beta$ and that the diagonal elements of the internal vielbein $e_i{}^j$ are related to the physical fields $\hat{h}^i{}_i$ by

$$e_i{}^i = e^{\hat{h}^i{}_i}. \tag{9.37}$$

The condition that the internal metric satisfies $det(G) = 1$ then leads to the relation $\sum_{i=d+1}^{10} \hat{h}^i{}_i = 0$.

We now proceed to compare the k_{IIB} and g group elements in $d < 9$ dimensions. Comparing the E_{n+1} parts of the group elements k_{IIB} and g we have

$$\begin{aligned} & e^{\left(\hat{h}^{d+1}{}_{d+1}+e_2\hat{\rho}\right)\hat{K}^{d+1}{}_{d+1}}...e^{\left(\hat{h}^9{}_9+\hat{e}_2\hat{\rho}\right)\hat{K}^9{}_9}e^{\left(h^{10}{}_{10}+\hat{e}_2\hat{\rho}\right)\hat{K}^{10}{}_{10}}e^{\hat{\sigma}\hat{R}} \\ =& e^{\dot{\varphi}_{d+1}\left(\hat{K}^{d+1}{}_{d+1}-\hat{K}^{d+2}{}_{d+2}\right)}...e^{\dot{\varphi}_9\left(-\frac{1}{4}\sum_{i=d+1}^{8}\hat{K}^i{}_i+\frac{3}{4}\left(\hat{K}^9{}_9+\hat{K}^{10}{}_{10}\right)\right)}e^{-2\dot{\varphi}_{10}\hat{R}}e^{\dot{\varphi}_{11}\left(\hat{K}^9{}_9-\hat{K}^{10}{}_{10}\right)}. \end{aligned} \tag{9.38}$$

Equating the coefficients of the generators gives

$$\begin{aligned} \hat{h}^{d+1}{}_{d+1} + \hat{e}_2\hat{\rho} &= \dot{\varphi}_{d+1} - \frac{1}{4}\dot{\varphi}_9, \\ \hat{h}^{d+2}{}_{d+2} + \hat{e}_2\hat{\rho} &= -\dot{\varphi}_{d+1} + \dot{\varphi}_{d+2} - \frac{1}{4}\dot{\varphi}_9, \\ &\vdots \\ h^8{}_8 + \hat{e}_2\hat{\rho} &= -\dot{\varphi}_7 + \dot{\varphi}_8 - \frac{1}{4}\dot{\varphi}_9, \\ h^9{}_9 + \hat{e}_2\hat{\rho} &= -\dot{\varphi}_8 + \frac{3}{4}\dot{\varphi}_9 + \dot{\varphi}_{11}, \\ h^{10}{}_{10} + \hat{e}_2\hat{\rho} &= \frac{3}{4}\dot{\varphi}_9 - \dot{\varphi}_{11}, \\ \hat{\sigma} &= -2\dot{\varphi}_{10}. \end{aligned} \tag{9.39}$$

Solving these equations for the E_{n+1} fields in terms of the scalars $\hat{\sigma}$, $\hat{h}^{d+1}{}_{d+1}, ..., \hat{h}^{10}{}_{10}$ and $\hat{\rho}$ found upon dimensional reduction yields

$$\begin{aligned} \dot{\varphi}_i &= \hat{h}^{d+1}{}_{d+1} + \hat{h}^{d+2}{}_{d+2} + ... + \hat{h}^i{}_i + (n-10+i)\,\frac{8}{8-n}\hat{e}_2\hat{\rho}, \quad d+1 \leq i < 8, \\ \dot{\varphi}_9 &= \frac{4n}{8-n}\hat{e}_2\hat{\rho}, \\ \dot{\varphi}_{10} &= -\frac{1}{2}\hat{\sigma} + \frac{2}{8-n}n\hat{e}_2\hat{\rho}, \\ \dot{\varphi}_{11} &= \sum_{a=d+1}^{9} h^a{}_a + \frac{4\,(n-2)}{8-n}. \end{aligned} \tag{9.40}$$

9.2.3 Type IIB Parameters

In Chevalley basis the E_{n+1} field $\dot{\varphi}_k$ is associated with node k of the E_{11} Dynkin diagram. The Volume of the torus in the type IIB theory $V_{n(B)}$, the d dimensional effective coupling g_d and the ratio of the radius r_{d+1} to the Planck length l_d are expressible in terms of the E_{n+1} fields $\dot{\varphi}_i$ in Chevalley basis, thus they are naturally associated with nodes of the E_{n+1} Dynkin diagram. For the volume of the IIB torus $V_{n(B)}$ one finds,

$$V_{n(B)} = \frac{r_{10}r_9...r_{d+1}}{l_{10}^n} = e^{n\hat{\beta}\hat{\rho}} = e^{\frac{8-n}{4}\dot{\varphi}_9}. \tag{9.41}$$

The effective coupling in d dimensions may be written

$$g_d = e^{\frac{8-n}{8}\hat{\phi} - \frac{n\hat{\beta}\hat{\rho}}{2}} = e^{-2\left(\frac{8-n}{8}\right)\dot{\varphi}_{10}}. \tag{9.42}$$

While the ratio of the radius of the circle in $d+1$ direction r_{d+1} to the d dimensional Planck length l_d is

$$\frac{r_{d+1}}{l_d} = \frac{l_{10}}{l_d}\frac{r_{d+1}}{l_{10}} = e^{\frac{8}{8-n}\hat{\beta}\hat{\rho} + \hat{h}^{d+1}{}_{d+1}} = e^{\dot{\varphi}_{d+1}}. \tag{9.43}$$

9.2.4 M-theory in $d = 11$

The Chevalley fields ϕ_i in the E_{11} group element for eleven dimensional supergravity are found as a function of the physical fields $h^i{}_i$ in section 7.3.

9.2.5 Dimensionally Reduced M-theory

The M-theory E_{11} group element k_M in the dimensionally reduced theory is found by deleting node d of the Dynkin diagram, this splits the E_{11} algebra into a gravity sub-algebra, associated with the generators of $SL(d)$ restricted to the Cartan subalgebra $K^a{}_a$, $a = 1, ..., d$ and a remaining internal sub-algebra, consisting of all the generators lying in the Cartan sub-algebra that are not elements of the gravity sub-algebra, these generate the E_m group. The generators associated with the gravity line are the $SL(d)$ generators, $K^a{}_b$, $a, b = 1, ...d$ while the generators $\dot{K}^i{}_j = K^i{}_j - \frac{1}{9-m}\delta^i_j \sum_{a=1}^d K^a{}_a$, for $i, j = d+1, ..., 11$ form the E_{n+1} internal subalgebra. One may verify that that the generators of the internal subalgebra satisfy $[P_a, \dot{K}^i{}_j] = 0$, for all $a = 1, ..., d$ and $i, j = d+1, ..., 11$, where P_a are the d dimensional spacetime momentum generators with commutation relations specified in equation (4.30). This decomposition allows us to write the group element k_M in the form

$$k_M = e^{\sum_{a=1}^d \dot{h}^a{}_a K^a{}_a} e^{\sum_{a=d+1}^{11} \dot{\varphi}_a T_a}. \tag{9.44}$$

where $\dot{\varphi}_a$ are the E_m Chevalley fields that parameterise the E_m Cartan subalgebra and

$$\begin{aligned} T_i &= \dot{K}^{i+1}{}_{i+1} - \dot{K}^{i+2}{}_{i+2} \quad for\ i = d+1, ..., 10, \\ T_{11} &= -\frac{1}{3}\left(\dot{K}^{d+1}{}_{d+1} + ... + \dot{K}^8{}_8\right) + \frac{2}{3}\left(\dot{K}^9{}_9 + \dot{K}^{10}{}_{10} + \dot{K}^{11}{}_{11}\right). \end{aligned} \tag{9.45}$$

Substituting the expression $\dot{K}^i{}_j = K^i{}_j - \frac{1}{9-m}\delta^i_j \sum_{a=1}^d K^a{}_a$ into equation (9.45) one finds that $H_i = T_i$ for $i = d+1, ..., 11$.

To find the E_m Chevalley fields parameterising k_M in the dimensionally reduced theory in terms of the physical fields that appear in $d = 11 - m$ dimensions we will compare the E_{11} group element k_M to another E_{11} group element g that corresponds to our chosen compactification ansatz (9.5).

The E_{11} group element that contains the M-theory physical fields in the dimensionally reduced theory after taking the metric compactification ansatz in equation (9.5) is given by

$$g = e^{x^\mu P_\mu} e^{h^a{}_a K^a{}_a + e_1 \rho \sum_{a=1}^{d} K^a{}_a} e^{h^i{}_i K^i{}_i + e_2 \rho \sum_{i=d+1}^{11} K^i{}_i}. \tag{9.46}$$

Evaluating the Cartan form gives,

$$g^{-1}\left(dx^a P_a + dx^i P_i\right) g = \sum_{a=1}^{d}\left(e^{h^a{}_a + e_1\rho}\right) dx^a P_a + \sum_{i=d+1}^{11}\left(e^{h^i{}_i + e_2\rho}\right) dx^i P_i, \tag{9.47}$$

where we have suppressed the same numerical factor for each $h^i{}_i$ given by the commutator of $K^i{}_i$ with the translation generators P_i and neglected terms involving derivatives of $h^i{}_i$. The Cartan form $\mathcal{V}$ in d dimensions, restricted to the Cartan sub-algebra, may be expanded as

$$\mathcal{V} \equiv dx^\mu \hat{e}_\mu{}^a P_a + dx^i \hat{e}_i{}^j P_j, \tag{9.48}$$

ignoring the terms in $K_a{}^b$. From our compactification ansatz, with the one-form gauge fields A set to zero, we may identify $\hat{e}_\mu{}^a = e^{\alpha\rho} e_\mu{}^a$ and $\hat{e}_j{}^i = e^{\beta\rho} e_j{}^i$. Upon comparing the Cartan form gdg^{-1} and the expansion of the Cartan form $\mathcal{V}$, with the above identifications of the vielbeins, we find

$$\sum_{a=1}^{d}\left(e^{h^a{}_a + e_1\rho}\right) dx^a P_a + \sum_{i=d+1}^{11}\left(e^{h^i{}_i + e_2\rho}\right) dx^i P_i = e^{\alpha\rho} e_\mu{}^a dx^\mu P_a + e^{\beta\rho} e_j{}^i dx^j P_i, \tag{9.49}$$

which gives $e_1 = \alpha$, $e_2 = \beta$. In addition, since we are only concerned with the Cartan subalgebra, upon comparing $\mathcal{V}$ and $g^{-1}dg$, we see that the diagonal elements of the internal vielbein $e_i{}^j$ are related to the physical fields $h^i{}_i$ by

$$e_i{}^i = e^{h^i{}_i}. \tag{9.50}$$

The condition that the internal metric satisfies $det(G) = 1$ then leads to the constraint $\sum_{i=d+1}^{11} h^i{}_i = 0$.

We now proceed to compare the k_M and g group elements in $d < 9$ dimensions. Comparing the E_m parts of the group elements k_M and g we have

$$\begin{aligned} &e^{\left(h^{d+1}{}_{d+1} + e_2\rho\right)K^{d+1}{}_{d+1}} \ldots e^{\left(h^9{}_9 + e_2\rho\right)K^9{}_9} e^{\left(h^{10}{}_{10} + e_2\rho\right)K^{10}{}_{10}} e^{\left(h^{11}{}_{11} + e_2\rho\right)K^{11}{}_{11}} \\ =& e^{\dot\varphi_{d+1}\left(K^{d+1}{}_{d+1} - K^{d+2}{}_{d+2}\right)} \ldots e^{\dot\varphi_{10}\left(K^{10}{}_{10} - K^{11}{}_{11}\right)} e^{\dot\varphi_{11}\left(\frac{2}{3}\sum_{a=d+1}^{11} K^a{}_a - \sum_{a=d+1}^{8} K^a{}_a\right)}. \end{aligned} \tag{9.51}$$

Equating the coefficients of the generators gives

$$\begin{aligned}
h^{d+1}{}_{d+1} + e_2\rho &= \dot\varphi_{d+1} - \frac{1}{3}\dot\varphi_{11}, \\
h^{d+2}{}_{d+2} + e_2\rho &= -\dot\varphi_{d+1} + \dot\varphi_{d+2} - \frac{1}{3}\dot\varphi_{11}, \\
&\dots \\
h^{8}{}_{8} + e_2\rho &= -\dot\varphi_{7} + \dot\varphi_{8} - \frac{1}{3}\dot\varphi_{11}, \\
h^{9}{}_{9} + e_2\rho &= -\dot\varphi_{8} + \dot\varphi_{9} + \frac{2}{3}\dot\varphi_{11}, \\
h^{10}{}_{10} + e_2\rho &= -\dot\varphi_{9} + \dot\varphi_{10} + \frac{2}{3}\dot\varphi_{11}, \\
h^{11}{}_{11} + e_2\rho &= -\dot\varphi_{10} + \frac{2}{3}\dot\varphi_{11}.
\end{aligned} \tag{9.52}$$

Solving these equations for the E_m fields in terms of the scalars $h^{d+1}{}_{d+1}, ..., h^{11}{}_{11}$ and ρ found upon dimensional reduction we find

$$\begin{aligned}
\dot\varphi_i &= h^{d+1}{}_{d+1} + h^{d+2}{}_{d+2} + ... + h^{i}{}_{i} + (m-11+i)\frac{9}{9-m}e_2\rho, \quad d+1 \le i < 8. \\
\dot\varphi_9 &= h^{d+1}{}_{d+1} + h^{d+2}{}_{d+2} + ... + h^{9}{}_{9} + \frac{6\,(m-3)}{9-m}e_2\rho, \\
\dot\varphi_{10} &= h^{d+1}{}_{d+1} + h^{d+2}{}_{d+2} + ... + h^{10}{}_{10} + \frac{3\,(m-3)}{9-m}e_2\rho, \\
\dot\varphi_{11} &= \frac{3m}{9-m}e_2\rho.
\end{aligned} \tag{9.53}$$

9.2.6 M-theory Parameters

Determining the fields $\dot\varphi_i$ in terms of the scalars $h^{d+1}{}_{d+1}, ..., h^{11}{}_{11}$ and ρ, in equation (9.53) allows one to express the M-theory parameters given in section 9.1, namely the dimensionless volume of the m torus V_m and the ratio of the radius r_{d+1} of a compact dimension to the d dimensional Planck length, in terms of the E_m fields $\dot\varphi_i$. Since the E_m fields $\dot\varphi_i$ are in Chevalley basis they are naturally associated with nodes of the E_m Dynkin diagram. For the volume of the M-theory torus V_m one finds,

$$V_m = \frac{r_{11}r_{10}r_9 ... r_{d+1}}{l_{11}^m} = e^{m\beta\rho} = e^{\frac{9-m}{3}\dot\varphi_{11}}. \tag{9.54}$$

While the ratio of the radius of the circle in the $d+1$ direction r_{d+1} to the d dimensional Planck length l_d is

$$\frac{r_{d+1}}{l_d} = \frac{l_{11}}{l_d}\frac{r_{d+1}}{l_{11}} = e^{\frac{9}{9-m}\beta\rho + h^{d+1}{}_{d+1}} = e^{\dot\varphi_{d+1}}. \tag{9.55}$$

9.2.7 Type IIA in $d = 10$

The Chevalley fields $\tilde{\phi}_i$ in the E_{11} group element for type IIA supergravity in $d = 10$ dimensions are found as a function of the physical fields $\tilde{h^i}_i$ and the type IIA dilaton $\tilde{\sigma}$ in section 7.3.

9.2.8 Dimensionally Reduced Type IIA

The type IIA E_{11} group element k_{IIA} in the dimensionally reduced theory is given by deleting node d of the Dynkin diagram, as in the eleven dimensional supergravity case, this splits the E_{11} algebra into a gravity subalgebra, associated with the generators of $SL(d)$ restricted to the Cartan subalgebra $\tilde{K}^a{}_a$, $a = 1, ..., d$ and a remaining internal sub-algebra, consisting of all the generators lying in the Cartan sub-algebra that are not elements of the gravity subalgebra, these generate the E_{n+1} group. The generators associated with the gravity line are the $SL(d)$ generators, $\tilde{K}^a{}_b$, $a, b = 1, ...d$, while the generators $\dot{K}^i{}_j = \tilde{K}^i{}_j - \frac{1}{8-n}\delta^i_j \sum_{a=1}^d \tilde{K}^a{}_a$, for $i, j = d+1, ..., 10$, and $\tilde{R}$ form the E_{n+1} internal subalgebra. Note that the generators $\dot{K}^i{}_j$ are not equal to those in the type IIB and eleven dimensional supergravity cases, in sections (9.2.2) and (9.2.5). The generators of the internal subalgebra satisfy $[P_a, \dot{K}^i{}_j] = 0$, for all $a = 1, ..., d$ and $i, j = d+1, ..., 10$, where P_a are the d dimensional spacetime translation generators with commutation relations specified in equation (4.30). The group element k_{IIA} may then be written in the form

$$k_{IIA} = e^{\sum_{a=1}^d \tilde{h}^a{}_a \tilde{K}^a{}_a} e^{\sum_{i=d+1}^{11} \dot{\varphi}_i \tilde{T}_i}, \tag{9.56}$$

where $\dot{\varphi}_a$ are the E_{n+1} Chevalley fields that parameterise the E_{n+1} Cartan subalgebra and

$$\begin{aligned}
\tilde{T}_a &= \dot{K}^a{}_a - \dot{K}^{a+1}{}_{a+1}, \quad a = d+1, ..., 9, \\
\tilde{T}_{10} &= -\frac{1}{8}\left(\dot{K}^{d+1}_{\ \ d+1} + ... + \dot{K}^9{}_9\right) + \frac{7}{8}\dot{K}^{10}{}_{10} - \frac{3}{2}\tilde{R}, \\
\tilde{T}_{11} &= -\frac{1}{4}\left(\dot{K}^{d+1}_{\ \ d+1} + ... + \dot{K}^8{}_8\right) + \frac{3}{4}\left(\dot{K}^9{}_9 + \dot{K}^{10}{}_{10}\right) + \tilde{R}.
\end{aligned} \tag{9.57}$$

Substituting the expression $\dot{K}^i{}_j = \tilde{K}^i{}_j - \frac{1}{8-n}\delta^i_j \sum_{a=1}^d \tilde{K}^a{}_a$ into equation (9.57) one finds $\tilde{H}_i = \tilde{T}_i$ for $i = d+1, ..., 11$.

As in the type IIA and eleven dimensional supergravity case, to find the E_{n+1} Chevalley fields parameterising k_{IIA} in the dimensionally reduced theory in terms of the physical fields that appear in $d = 10 - n$ dimensions we compare the E_{11} group element k_{IIA} to the E_{11} group element g that corresponds to our chosen compactification ansatz (9.5).

The E_{11} group element that contains the type IIA physical fields in the dimensionally reduced theory after taking the metric compactification ansatz in equation (9.5) is given by

$$g = e^{x^\mu P_\mu} e^{\tilde{h}^a{}_a \tilde{K}^a{}_a + e_1 \tilde{\rho} \sum_{a=1}^d \tilde{K}^a{}_a} e^{\tilde{h}^i{}_i \tilde{K}^i{}_i + e_2 \tilde{\rho} \sum_{i=d+1}^{10} \tilde{K}^i{}_i} e^{\tilde{R}\hat{\tilde{\sigma}}}. \tag{9.58}$$

Evaluating the Cartan form gives,

$$g^{-1}\left(dx^a P_a + dx^i P_i\right) g = \sum_{a=1}^{d}\left(e^{\tilde{h}^a{}_a + e_1\tilde{\rho}}\right) dx^a P_a + \sum_{i=d+1}^{10}\left(e^{\tilde{h}^i{}_i + \tilde{e}_2\tilde{\rho}}\right) dx^i P_i, \tag{9.59}$$

again up to the same numerical for each $\tilde{h}^i{}_i$. The Cartan form $\mathcal{V}$ in d dimensions, restricted to the Cartan sub-algebra, may be expanded as

$$\mathcal{V} \equiv dx^\mu \hat{e}_\mu{}^a P_a + dx^i \hat{e}_i{}^j P_j. \tag{9.60}$$

We may then identify $\hat{e}_\mu{}^a = e^{\tilde{\alpha}\tilde{\rho}} e_\mu{}^a$ and $\hat{e}_j{}^i = e^{\tilde{\beta}\tilde{\rho}} e_j{}^i$. Comparing the Cartan form gdg^{-1} and the expansion of the Cartan form $\mathcal{V}$, with the above identifications of the vielbeins, we find

$$\sum_{a=1}^{d}\left(e^{\tilde{h}^a{}_a + \tilde{e}_1\tilde{\rho}}\right) dx^a P_a + \sum_{i=d+1}^{10}\left(e^{\tilde{h}^i{}_i + \tilde{e}_2\tilde{\rho}}\right) dx^i P_i = e^{\tilde{\alpha}\tilde{\rho}} e_\mu{}^a dx^\mu P_a + e^{\tilde{\beta}\tilde{\rho}} e_j{}^i dx^j P_i. \tag{9.61}$$

So we see that $\tilde{e}_1 = \tilde{\alpha}$, $\tilde{e}_2 = \tilde{\beta}$ and that the diagonal elements of the internal vielbein $e_i{}^j$ are related to the physical fields $h^i{}_i$ by

$$e_i{}^i = e^{h^i{}_i}. \tag{9.62}$$

The condition that the internal metric satisfies $det(G) = 1$ then leads to the relation $\sum_{i=d+1}^{11} h^i{}_i = 0$.

We now proceed to compare the k_{IIA} and g group elements in $d < 9$ dimensions. Comparing the E_{n+1} parts of the group elements k_{IIA} and g we have

$$\begin{aligned}
&e^{\left(\tilde{h}^{d+1}{}_{d+1} + \tilde{e}_2\tilde{\rho}\right)\tilde{K}^{d+1}{}_{d+1}} \ldots e^{\left(\tilde{h}^9{}_9 + \tilde{e}_2\tilde{\rho}\right)\tilde{K}^9{}_9} e^{\left(\tilde{h}^{10}{}_{10} + \tilde{e}_2\tilde{\rho}\right)\tilde{K}^{10}{}_{10}} e^{\tilde{\sigma}\tilde{R}} \\
=& e^{\dot{\varphi}_{d+1}\left(\tilde{K}^{d+1}{}_{d+1} - \tilde{K}^{d+2}{}_{d+2}\right)} \ldots e^{\dot{\varphi}_8\left(\tilde{K}^8{}_8 - \tilde{K}^9{}_9\right)} e^{\dot{\varphi}_9\left(\tilde{K}^9{}_9 - \tilde{K}^{10}{}_{10}\right)} e^{\dot{\varphi}_{10}\left(-\frac{1}{8}\left(\tilde{K}^{d+1}{}_{d+1} + \ldots + \tilde{K}^9{}_9\right) + \frac{7}{8}\tilde{K}^{10}{}_{10} - \frac{3}{2}\tilde{R}\right)} \times \\
&e^{\dot{\varphi}_{11}\left(-\frac{1}{4}\left(\tilde{K}^{d+1}{}_{d+1} + \ldots + \tilde{K}^8{}_8\right) + \frac{3}{4}\left(\tilde{K}^9{}_9 + \tilde{K}^{10}{}_{10}\right) + \tilde{R}\right)}.
\end{aligned} \tag{9.63}$$

Equating the coefficients of the generators we find

$$\begin{aligned}
\tilde{h}^{d+1}{}_{d+1} + \tilde{e}_2\tilde{\rho} &= \dot{\varphi}_{d+1} - \frac{1}{8}\dot{\varphi}_{10} - \frac{1}{4}\dot{\varphi}_{11}, \\
\tilde{h}^{d+2}{}_{d+2} + \tilde{e}_2\tilde{\rho} &= -\dot{\varphi}_{d+1} + \dot{\varphi}_{d+2} - \frac{1}{8}\dot{\varphi}_{10} - \frac{1}{4}\dot{\varphi}_{11}, \\
&\vdots \\
\tilde{h}^{8}{}_{8} + \tilde{e}_2\tilde{\rho} &= -\dot{\varphi}_{7} + \dot{\varphi}_{8} - \frac{1}{8}\dot{\varphi}_{10} - \frac{1}{4}\dot{\varphi}_{11}, \\
\tilde{h}^{9}{}_{9} + \tilde{e}_2\tilde{\rho} &= -\dot{\varphi}_{8} + \dot{\varphi}_{9} - \frac{1}{8}\dot{\varphi}_{10} + \frac{3}{4}\dot{\varphi}_{11}, \\
\tilde{h}^{10}{}_{10} + \tilde{e}_2\tilde{\rho} &= -\dot{\varphi}_{9} + \frac{7}{8}\dot{\varphi}_{10} + \frac{3}{4}\dot{\varphi}_{11}, \\
\tilde{\sigma} &= -\frac{3}{2}\dot{\varphi}_{10} + \dot{\varphi}_{11}.
\end{aligned} \tag{9.64}$$

Solving these equations for the IIA E_{11} fields $\dot{\varphi}_i$ in terms of the scalars $\tilde{h}^{d+1}{}_{d+1}, \tilde{h}^{d+2}{}_{d+2}, ..., \tilde{h}^{10}{}_{10}, \tilde{\sigma}, \tilde{\rho}$ found upon dimensional reduction we find

$$\begin{aligned}
\dot{\varphi}_i &= \tilde{h}^{d+1}{}_{d+1} + \tilde{h}^{d+2}{}_{d+2} + ... + \tilde{h}^{i}{}_{i} + (n-10+i)\frac{8}{8-n}\tilde{e}_2\tilde{\rho}, \quad for\ i = d+1, ..., 8. \\
\dot{\varphi}_9 &= \tilde{h}^{d+1}{}_{d+1} + \tilde{h}^{d+2}{}_{d+2} + ... + \tilde{h}^{9}{}_{9} + \frac{5n-8}{8-n}\tilde{e}_2\tilde{\rho} - \frac{1}{4}\tilde{\sigma}, \\
\dot{\varphi}_{10} &= -\frac{1}{2}\tilde{\sigma} + \frac{2}{8-n}n\tilde{e}_2\tilde{\rho}, \\
\dot{\varphi}_{11} &= \frac{1}{4}\tilde{\sigma} + \frac{3}{8-n}n\tilde{e}_2\tilde{\rho}.
\end{aligned} \tag{9.65}$$

9.2.9 Type IIA Parameters

We may express the parameters $V_{n(A)}$, g_d and r_{d+1} in terms of the E_{n+1} fields $\dot{\varphi}_i$ in Chevalley basis, where they are then naturally associated with nodes of the E_{11} Dynkin diagram. For the volume of the IIA torus $V_{n(A)}$ one finds,

$$V_{n(A)} = \frac{r_{10}r_9...r_{d+1}}{l_{10}^n} = e^{n\tilde{\beta}\tilde{\rho}} = e^{\frac{8-n}{8}(\dot{\varphi}_{10}+2\dot{\varphi}_{11})}. \tag{9.66}$$

The effective coupling in d dimensions may be written

$$g_d = e^{\frac{8-n}{8}\sigma - \frac{n\beta\tilde{\rho}}{2}} = e^{-2\left(\frac{8-n}{8}\right)\dot{\varphi}_{10}}. \tag{9.67}$$

While the ratio of the radius of the circle in the $d+1$ direction r_{d+1} to the d dimensional Planck length l_d is

$$\frac{r_{d+1}}{l_d} = \frac{l_{10}}{l_d}\frac{r_{d+1}}{l_{10}} = e^{\frac{8}{8-n}\beta\tilde{\rho} + \tilde{h}^{d+1}{}_{d+1}} = e^{\dot{\varphi}_{d+1}}. \tag{9.68}$$

9.3 Relationships Between IIA, IIB and M-theory

After compactification on a torus to $d < 10$ dimensions type IIA/B string theory and M-theory are equivalent and there is a one to one correspondence between the fields of these theories. To derive an expression for the fields in one theory in terms of those of another we equate the group elements containing the physical fields, which we have denoted g, from each theory and expand the physical generators of one theory in terms of the physical generators of the other, the correspondence between the physical fields of the two theories may then be read off from the coefficients of the physical generators. The relationship between the physical generators of two theories is obtained by equating their Chevalley generators. We are interested in the dilatonic scalars that appear as the physical fields in each theory after dimensional reduction to $d = 10 - n$ or $d = 11 - m$ dimensions, although one need not dimensionally reduce these theories for the correspondence to hold. So we will again restrict the E_{11} group element to that generated by the Cartan subalgebra elements H_a, $a = 1, .., 11$.

9.3.1 M-theory and IIA

Equating the Chevalley generators of M-theory and IIA one has [87]

$$\begin{aligned} K^a{}_a &= \tilde{K}^a{}_a, \quad a = 1, ..., d, \\ K^i{}_i &= \tilde{K}^i{}_i, \quad i = d+1, ..., 10 \\ K^{11}{}_{11} &= \frac{1}{8}\sum_{i=1}^{10} \tilde{K}^i{}_i + \frac{3}{2}\tilde{R}. \end{aligned} \tag{9.69}$$

The M-theory group element g_M containing the physical fields is

$$g_M = e^{\sum_{a=1}^{d} h^a{}_a K^a{}_a + e_1\rho \sum_{a=1}^{d} K^a{}_a} e^{\sum_{i=d+1}^{11} h^i{}_i K^i{}_i + e_2\rho\sum_{i=d+1}^{11} K^i{}_i}. \tag{9.70}$$

Using the above relations this may be written in terms of the IIA physical generators, one finds

$$\begin{aligned} g_M = {} & e^{\sum_{a=1}^{d} h^a{}_a \tilde{K}^a{}_a + e_1\rho\sum_{a=1}^{d}\tilde{K}^a{}_a} \times \\ & e^{\sum_{i=d+1}^{10} h^i{}_i \tilde{K}^i{}_i + h^{11}{}_{11}\left(\frac{1}{8}\sum_{i=1}^{10}\tilde{K}^i{}_i + \frac{3}{2}\tilde{R}\right) + e_2\rho\left(\frac{9}{8}\sum_{i=1}^{10}\tilde{K}^i{}_i + \frac{3}{2}\tilde{R}\right)}. \end{aligned} \tag{9.71}$$

Equating the M-theory E_{11} group element g_M parameterised by the M-theory physical fields in d dimensions with the IIA E_{11} group element g_{IIA} parameterised by the IIA physical fields in d dimensions gives

$$\begin{aligned} & e^{\sum_{a=1}^{d} h^a{}_a \tilde{K}^a{}_a + e_1\rho\sum_{a=1}^{d}\tilde{K}^a{}_a} e^{\sum_{i=d+1}^{10} h^i{}_i \tilde{K}^i{}_i + h^{11}{}_{11}\left(\frac{1}{8}\sum_{i=1}^{10}\tilde{K}^i{}_i + \frac{3}{2}\tilde{R}\right) + e_2\rho\left(\frac{9}{8}\sum_{i=1}^{10}\tilde{K}^i{}_i + \frac{3}{2}\tilde{R}\right)} \\ & \qquad = e^{\sum_{a=1}^{d} \tilde{h}^a{}_a \tilde{K}^a{}_a + \tilde{e}_1\tilde{\rho}\sum_{a=1}^{d}\tilde{K}^a{}_a} e^{\sum_{i=d+1}^{10} \tilde{h}^i{}_i \tilde{K}^i{}_i + e_2\tilde{\rho}\sum_{i=d+1}^{10}\tilde{K}^i{}_i} e^{\tilde{\sigma}\tilde{R}}. \end{aligned} \tag{9.72}$$

The correspondence between the M-theory and IIA physical fields may then be determined by examining the coefficients of the physical generators in this equation. One finds

$$\begin{aligned}
\tilde{h}^a{}_a + \tilde{e}_2\tilde{\rho} &= h^a{}_a + e_2\rho + \frac{1}{8}\left(h^{11}{}_{11} + e_2\rho\right), \quad a = 1, ..., d, \\
\tilde{h}^i{}_i + \tilde{e}_2\tilde{\rho} &= h^i{}_i + \frac{1}{8}h^{11}{}_{11} + \frac{9}{8}e_2\rho, \quad i = d+1, ..., 10, \\
\tilde{\sigma} &= \frac{3}{2}\left(h^{11}{}_{11} + e_2\rho\right).
\end{aligned} \tag{9.73}$$

We may solve these equations to write the volume of the IIA torus $V_{n(A)}$ as a function of the M-theory physical fields and the volume of the M-theory torus V_m as a function of the IIA physical fields, this gives

$$V_{n(A)} = e^{n\tilde{e}_2\tilde{\rho}} = e^{\frac{8-n}{8}\sum_{i=d+1}^{10} h^i{}_i + \frac{9}{8}ne_2\rho} \tag{9.74}$$

and

$$V_m = e^{(n+1)e_2\rho} = e^{n\tilde{e}_2\tilde{\rho} + \frac{1}{3}\left(\frac{8-n}{4}\right)\tilde{\sigma}}. \tag{9.75}$$

These relations may be simplified by writing the physical fields in terms of the M-theory and IIA E_{n+1} Chevalley fields, equation (9.74) becomes

$$V_{n(A)} = e^{\frac{8-n}{8}(\dot{\varphi}_{10} + 2\dot{\varphi}_{11})}, \tag{9.76}$$

while equation (9.75) becomes

$$V_m = e^{\frac{8-n}{3}\dot{\varphi}_{11}}. \tag{9.77}$$

We also find that the last equation of (9.73) is the standard relationship between the IIA coupling $g_{s(A)}$ in ten dimensions, the eleven dimensional Planck length l_{11} and the radius of the compactified eleventh dimension r_{11}, since $g_{s(A)} = e^{\tilde{\sigma}}$

$$\begin{aligned}
g_{s(A)}^{\frac{2}{3}} &= e^{\frac{2}{3}\tilde{\sigma}} \\
&= e^{\left(h^{11}{}_{11} + e_2\rho\right)} \\
&= \frac{r_{11}}{l_{11}}.
\end{aligned} \tag{9.78}$$

9.3.2 IIA and IIB

Equating the Chevalley generators of type IIA and type IIB gives [87]

$$\begin{aligned} \hat{K}^a{}_a &= \tilde{K}^a{}_a, \quad a = 1, ..., d, \\ \hat{K}^i{}_i &= \tilde{K}^i{}_i, \quad i = d+1, ..., 9, \\ \hat{K}^{10}{}_{10} &= \frac{1}{4}\sum_{i=1}^{9}\tilde{K}^i{}_i - \frac{3}{4}\tilde{K}^{10}{}_{10} - \tilde{R}, \\ \hat{R} &= \frac{1}{16}\sum_{i=1}^{9}\tilde{K}^i{}_i - \frac{7}{16}\tilde{K}^{10}{}_{10} + \frac{3}{4}\tilde{R}. \end{aligned} \tag{9.79}$$

The IIB group element g_{IIB} containing the physical fields is

$$g_{IIB} = e^{\sum_{a=1}^{d}\hat{h}^a{}_a\hat{K}^a{}_a + e_1\hat{\rho}\sum_{a=1}^{d}\hat{K}^a{}_a} e^{\sum_{i=d+1}^{10}\hat{h}^i{}_i\hat{K}^i{}_i + e_2\hat{\rho}\sum_{i=d+1}^{10}\hat{K}^i{}_i} e^{\hat{\sigma}\hat{R}}. \tag{9.80}$$

Using the above relations this may be written in terms of the IIA physical generators, one finds

$$\begin{aligned} g_{IIB} = {} & e^{\sum_{a=1}^{d}\hat{h}^a{}_a\tilde{K}^a{}_a + \hat{e}_1\hat{\rho}\sum_{a=1}^{d}\tilde{K}^a{}_a} \times \\ & e^{\sum_{i=d+1}^{9}\hat{h}^i{}_i\tilde{K}^i{}_i + \hat{h}^{10}{}_{10}\left(\frac{1}{4}\sum_{i=1}^{9}\tilde{K}^i{}_i - \frac{3}{4}\tilde{K}^{10}{}_{10} - \tilde{R}\right) + \hat{e}_2\hat{\rho}\left(\frac{5}{4}\sum_{i=1}^{9}\tilde{K}^i{}_i - \frac{3}{4}\tilde{K}^{10}{}_{10} - \tilde{R}\right)} \times \\ & e^{\hat{\sigma}\left(\frac{1}{16}\sum_{i=1}^{9}\tilde{K}^i{}_i - \frac{7}{16}\tilde{K}^{10}{}_{10} + \frac{3}{4}\tilde{R}\right)}. \end{aligned} \tag{9.81}$$

Equating the IIB E_{11} group element g_{IIB} parameterised by the IIB physical fields in d dimensions with the IIA E_{11} group element g_{IIA} parameterised by the IIA physical fields in d dimensions gives

$$\begin{aligned} & e^{\sum_{a=1}^{d}\hat{h}^a{}_a\tilde{K}^a{}_a + \hat{e}_1\hat{\rho}\sum_{a=1}^{d}\tilde{K}^a{}_a} e^{\sum_{i=d+1}^{9}\hat{h}^i{}_i\tilde{K}^i{}_i + \hat{h}^{10}{}_{10}\left(\frac{1}{4}\sum_{i=1}^{9}\tilde{K}^i{}_i - \frac{3}{4}\tilde{K}^{10}{}_{10} - \tilde{R}\right) + \hat{e}_2\hat{\rho}\left(\frac{5}{4}\sum_{i=1}^{9}\tilde{K}^i{}_i - \frac{3}{4}\tilde{K}^{10}{}_{10} - \tilde{R}\right)} \times \\ & e^{\hat{\sigma}\left(\frac{1}{16}\sum_{i=1}^{9}\tilde{K}^i{}_i - \frac{7}{16}\tilde{K}^{10}{}_{10} + \frac{3}{4}\tilde{R}\right)} = e^{\sum_{a=1}^{d}\tilde{h}^a{}_a\tilde{K}^a{}_a + \tilde{e}_1\tilde{\rho}\sum_{a=1}^{d}\tilde{K}^a{}_a} e^{\sum_{i=d+1}^{10}\tilde{h}^i{}_i\tilde{K}^i{}_i + e_2\tilde{\rho}\sum_{i=d+1}^{10}\tilde{K}^i{}_i} e^{\tilde{\sigma}\tilde{R}}. \end{aligned} \tag{9.82}$$

The correspondence between the IIB and IIA physical fields may then be determined by examining the coefficients of the physical generators in this equation, this results in the relations

$$\begin{aligned} \tilde{h}^a{}_a + \tilde{e}_2\tilde{\rho} &= \hat{h}^a{}_a + \hat{e}_2\hat{\rho} + \frac{1}{4}\left(\hat{h}^{10}{}_{10} + \hat{e}_2\hat{\rho}\right), \quad a = 1, ..., d, \\ \tilde{h}^i{}_i + \tilde{e}_2\tilde{\rho} &= \hat{h}^i{}_i + \frac{1}{4}\hat{h}^{10}{}_{10} + \frac{5}{4}\hat{e}_2\hat{\rho} + \frac{1}{16}\hat{\sigma}, \quad i = d+1, ..., 9, \\ \tilde{h}^{10}{}_{10} + \tilde{e}_2\tilde{\rho} &= -\frac{3}{4}\left(\hat{h}^{10}{}_{10} + \hat{e}_2\hat{\rho}\right) - \frac{7}{16}\hat{\sigma}, \\ \tilde{\sigma} &= -\left(\hat{h}^{10}{}_{10} + \hat{e}_2\hat{\rho}\right) + \frac{3}{4}\hat{\sigma}. \end{aligned} \tag{9.83}$$

We will again solve these equations to write the volume of the IIA torus $V_{n(A)}$ as a function of the IIB physical fields and the IIB torus $V_{n(B)}$ as a function of the IIA physical fields, this gives

$$V_{n(A)} = e^{n\tilde{e}_2\tilde{\rho}} = e^{\frac{8-n}{4}\sum_{i=d+1}^{9}\hat{h}^i{}_i + \frac{5n-8}{4}n\hat{e}_2\hat{\rho} + \frac{n-8}{16}\hat{\sigma}}, \tag{9.84}$$

and

$$V_{n(B)} = e^{n\hat{e}_2\hat{\rho}} = e^{\frac{8-n}{4}\sum_{i=d+1}^{9}\tilde{h}^i{}_i + \frac{5n-8}{4}\tilde{e}_2\tilde{\rho} + \left(\frac{n-8}{16}\right)\tilde{\sigma}}. \tag{9.85}$$

These relations may be simplified by writing the physical fields in terms of the IIA and IIB Chevalley fields, equation (9.84) becomes

$$V_{n(A)} = e^{\frac{8-n}{8}(\dot{\varphi}_{10}+2\dot{\varphi}_{11})}, \tag{9.86}$$

while equation (9.85) becomes

$$V_{n(B)} = e^{\frac{8-n}{4}\dot{\varphi}_9}. \tag{9.87}$$

The last two equations of (9.83) give the T-duality correspondence between the radius of the circle upon which the IIA theory is compactified $\tilde{r}_{10}$ and the radius of the circle upon which the IIB theory is compactified $\hat{r}_{10}$ and the T-duality correspondence between the coupling constants $g_{s(IIA)}$ and $g_{s(IIB)}$. The T-duality correspondence between the radii is given by

$$\begin{aligned}
\tilde{r}_{10} &= e^{\tilde{h}^{10}{}_{10}+\tilde{e}_2\tilde{\rho}} l_{10(A)} \\
&= e^{\tilde{h}^{10}{}_{10}+\tilde{e}_2\tilde{\rho}} l_s e^{\frac{\tilde{\sigma}}{4}} \\
&= e^{-\frac{3}{4}\left(\hat{h}^{10}{}_{10}+\hat{e}_2\hat{\rho}\right)-\frac{7}{16}\hat{\sigma}} e^{-\frac{1}{4}\left(\hat{h}^{10}{}_{10}+\hat{e}_2\hat{\rho}\right)+\frac{3}{16}\hat{\sigma}} l_s \\
&= e^{-\left(\hat{h}^{10}{}_{10}+\hat{e}_2\hat{\rho}\right)} e^{-\frac{1}{4}\hat{\sigma}} l_s \\
&= \frac{l_{10(B)}}{\hat{r}_{10}} e^{-\frac{1}{4}\hat{\sigma}} l_s \\
&= \frac{l_s^2}{\hat{r}_{10}}.
\end{aligned} \tag{9.88}$$

While the T-duality correspondence between the coupling constants is given by

$$\begin{aligned}
g_{s(A)} &= e^{\tilde{\sigma}} \\
&= e^{-\left(\hat{h}^{10}{}_{10}+\hat{e}_2\hat{\rho}\right)+\frac{3}{4}\hat{\sigma}} \\
&= \frac{l_{10(B)}}{\hat{r}_{10}} e^{\frac{3}{4}\hat{\sigma}} \\
&= \frac{l_s}{\hat{r}_{10}} g_{s(B)}.
\end{aligned} \tag{9.89}$$

Note that these relations agree with the Buscher rules for T-duality that relate the string coupling $g_{s(A)}$ and radius $\tilde{r}$ of the type IIA string to the string coupling $g_{s(B)}$ and radius $\hat{r}$ of the type IIB string.

9.3.3 M-theory and IIB

Equating the Chevalley generators of IIB and M-theory one has [87]

$$\begin{aligned}
\hat{K}^a{}_a &= K^a{}_a, \quad a = 1, ..., d, \\
\hat{K}^i{}_i &= K^i{}_i, \quad i = d+1, ..., 9, \\
\hat{K}^{10}{}_{10} &= \frac{1}{3}\sum_{i=1}^{9} K^i{}_i - \frac{2}{3}\left(K^{10}{}_{10} + K^{11}{}_{11}\right), \\
\hat{R} &= -\frac{1}{2}\left(K^{10}{}_{10} - K^{11}{}_{11}\right).
\end{aligned} \tag{9.90}$$

The type IIB group element g_{IIB} containing the physical fields is

$$g_{IIB} = e^{\sum_{a=1}^{d} \hat{h}^a{}_a \hat{K}^a{}_a + \hat{e}_1\hat{\rho}\sum_{a=1}^{d}\hat{K}^a{}_a} e^{\sum_{i=d+1}^{10} \hat{h}^i{}_i \hat{K}^i{}_i + \hat{e}_2\hat{\rho}\sum_{i=d+1}^{10}\hat{K}^i{}_i} e^{\hat{\sigma}\hat{R}}. \tag{9.91}$$

Using the above relations this may be written in terms of the M-theory physical generators, one finds

$$\begin{aligned}
g_{IIB} = &\, e^{\sum_{a=1}^{d} \hat{h}^a{}_a K^a{}_a + \hat{e}_1\hat{\rho}\sum_{a=1}^{d} K^a{}_a} \times \\
&\, e^{\sum_{i=d+1}^{9} \hat{h}^i{}_i K^i{}_i + \hat{h}^{10}{}_{10}\left(\frac{1}{3}\sum_{i=1}^{9} K^i{}_i - \frac{2}{3}\left(K^{10}{}_{10}+K^{11}{}_{11}\right)\right) + \hat{e}_2\hat{\rho}\left(\frac{4}{3}\sum_{i=1}^{9} K^i{}_i - \frac{2}{3}\left(K^{10}{}_{10}+K^{11}{}_{11}\right)\right)} \\
&\times e^{\hat{\sigma}\left(-\frac{1}{2}\left(K^{10}{}_{10}-K^{11}{}_{11}\right)\right)}.
\end{aligned} \tag{9.92}$$

Equating the IIB E_{11} group element g_{IIB} parameterised by the IIB physical fields in d dimensions with the M-theory E_{11} group element g_M parameterised by the eleven dimensional supergravity physical fields in d dimensions gives

$$\begin{aligned}
&e^{\sum_{a=1}^{d} \hat{h}^a{}_a K^a{}_a + e_1\hat{\rho}\sum_{a=1}^{d} K^a{}_a} e^{\sum_{i=d+1}^{9} \hat{h}^i{}_i K^i{}_i + \hat{h}^{10}{}_{10}\left(\frac{1}{3}\sum_{i=1}^{9} K^i{}_i - \frac{2}{3}\left(K^{10}{}_{10}+K^{11}{}_{11}\right)\right)} \times \\
&e^{\hat{e}_2\hat{\rho}\left(\frac{4}{3}\sum_{i=1}^{9} K^i{}_i - \frac{2}{3}\left(K^{10}{}_{10}+K^{11}{}_{11}\right)\right)} e^{\hat{\sigma}\left(-\frac{1}{2}\left(K^{10}{}_{10}-K^{11}{}_{11}\right)\right)} \\
&= e^{\sum_{a=1}^{d} h^a{}_a K^a{}_a + e_1\rho\sum_{a=1}^{d} K^a{}_a} e^{\sum_{i=d+1}^{11} h^i{}_i K^i{}_i + e_2\rho\sum_{i=d+1}^{11} K^i{}_i}.
\end{aligned} \tag{9.93}$$

The correspondence between the M-theory and IIB physical fields may then be determined by examining the coefficients of the physical generators in this equation. One finds

$$\begin{aligned}
h^a{}_a + e_1\rho &= \hat{h}^a{}_a + \hat{e}_1\hat{\rho} + \frac{1}{3}\left(\hat{h}^{10}{}_{10} + \hat{e}_2\hat{\rho}\right), \quad a = 1, ..., d, \\
h^i{}_i + e_2\rho &= \hat{h}^i{}_i + \frac{4}{3}\hat{e}_2\rho + \frac{1}{3}\hat{h}^{10}{}_{10}, \quad i = d+1, ..., 9, \\
h^{10}{}_{10} + e_2\rho &= -\frac{2}{3}\hat{h}^{10}{}_{10} - \frac{2}{3}\hat{e}_2\hat{\rho} - \frac{1}{2}\hat{\sigma}, \\
h^{11}{}_{11} + e_2\rho &= -\frac{2}{3}\hat{h}^{10}{}_{10} - \frac{2}{3}\hat{e}_2\hat{\rho} + \frac{1}{2}\hat{\sigma}.
\end{aligned} \tag{9.94}$$

Again, these may be used to derive useful relations between for the volume of the M-theory torus V_m as a function of the IIB physical fields amd volume of the IIB torus $V_{n(B)}$ as a function of the M-theory physical fields, we find

$$V_m = e^{(n+1)e_2\rho} = e^{\frac{4}{3}(n-2)\hat{e}_2\hat{\rho} - \frac{n-8}{3}\hat{h}^{10}{}_{10}} \tag{9.95}$$

and

$$V_{n(B)} = e^{n\hat{e}_2\hat{\rho}} = e^{\frac{8-n}{4}\sum_{i=d+1}^{9} h^i{}_i + \frac{3(n-2)}{2}e_2\rho}. \tag{9.96}$$

In terms of the IIB Chevalley fields the volume of the M-theory torus V_m may be written

$$V_m = e^{\frac{8-n}{3}\dot{\varphi}_{11}}, \tag{9.97}$$

while the IIB torus $V_{n(B)}$ as a function of the M-theory Chevalley fields is

$$V_{n(B)} = e^{\frac{8-n}{4}\dot{\varphi}_9}. \tag{9.98}$$

9.4 Limits

An arbitrary higher derivative term in $d < 10$ dimensions is a polynomial in the curvature R, Cartan forms S or field strengths F with an associated coefficient function that transforms as an $E_{n+1}(\mathbb{Z})$ automorphic form. Strong constraints on the structure of this E_{n+1} automorphic form may be found by examining the limits of these higher derivative term in the various physical parameters of type IIA/B string theory and M-theory. We will examine the constraints imposed on these higher derivative terms by taking the large volume limits of the IIA torus $V_{n(A)}$, IIB torus $V_{n(B)}$, M-theory torus V_m, the decompactification of a single dimension of radius r_{d+1} and the perturbative or weak coupling limit where the effective coupling in d dimensions g_d tends to zero.

Upon compactification on a torus to $d < 10$ dimensions type IIA/B string theory and M-theory are equivalent. Therefore one would expect that these limits should be independent of whether we choose to parameterise the d dimensional theory by type IIA, type IIB or M-theory physical fields. Our identification of the parameters involved in taking these limits in terms of the E_{n+1} Chevalley fields shows that this is indeed the case. From equations (9.66), (9.76) and (9.86) we see that the volume of the IIA torus $V_{n(A)}$ may be written in terms of the E_{n+1} Chevalley fields, in each theory we find $V_{n(A)} = e^{\frac{8-n}{n}(\dot{\varphi}_{10}+2\dot{\varphi}_{11})}$ where $\dot{\varphi}_{10}$ and $\dot{\varphi}_{11}$ are the E_{n+1} Chevalley fields, thus, the IIA volume limit is associated with nodes 10 and 11 of the E_{n+1} Dynkin diagram found after deleting node d in the E_{11} Dynkin diagram, regardless of whether we choose to parameterise the d dimensional theory in terms of the type IIA, type IIB or M-theory physical fields. Similarly, the volume of the type IIB torus $V_{n(B)}$ may be written $V_{n(B)} = e^{\frac{8-n}{4}\dot{\varphi}_9}$, so the type IIB volume limit is associated with node 9 of the E_{n+1} Dynkin diagram, found after deleting node d in the E_{11} Dynkin diagram. The remaining parameters of interest to us, the volume of the M-theory torus V_m, the ratio of the radius of the circle in the $(d+1)$ direction to the d dimensional Planck length l_d, and the effective coupling in d dimensions g_d are also associated with various nodes of the E_{n+1} Dynkin diagram. One finds $V_m = e^{\frac{8-n}{3}\dot{\varphi}_{11}}$, $\frac{r_{d+1}}{l_d} = e^{\dot{\varphi}_{d+1}}$ and $g_d = e^{-2(\frac{8-n}{8})\dot{\varphi}_{10}}$, so the volume of the M-theory torus is associated with node 11, the ratio of the radius of the circle in the $(d+1)$ direction to the d dimensional Planck length l_d is associated with node $(d+1)$, while the effective coupling in d dimensions is associated with node 10.

We will now examine the constraints placed on an arbitrary higher derivative term in d dimensions by taking each of these limits. Note that one must be careful when considering non-analytic terms in the action that appear divergent in a given limit.

9.4.1 IIB Volume Limit

Type IIB string theory in $d = 10$ dimensions exhibits an $SL(2,\mathbb{Z})$ symmetry. So an arbitrary higher derivative term in $d = 10 - n$ dimensions should, in the large volume limit $V_n \to \infty$ give a sum of $d = 10$ higher derivative terms whose coefficient functions are $SL(2,\mathbb{Z})$ automorphic forms. The volume of the IIB torus $V_{n(B)}$ may be expressed in terms of the E_{n+1} Chevalley fields, one finds

$$V_{n(B)} = \frac{r_{10}r_9 \ldots r_{d+1}}{l_{10}^n} = e^{n\beta\rho} = e^{\frac{8-n}{4}\dot{\varphi}_9}. \tag{9.99}$$

So the large volume limit $V_{n(B)} \to \infty$ in the IIB theory is equivalent to $\dot{\varphi}_9 \to \infty$. Taking $\dot{\varphi}_9 \to \infty$ corresponds to deleting node n in the E_{n+1} Dynkin diagram, as shown in figure 18. This decomposes

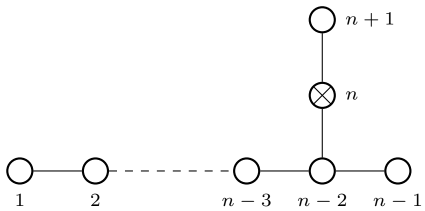

Figure 18: Dynkin diagram for E_{n+1} with node n deleted

the E_{n+1} algebra in terms of a $GL(1) \times SL(2) \times SL(n)$ subalgebra. It then follows that taking the large volume limit $V_n \to \infty$ splits an E_{n+1} automorphic form into a sum of $GL(1) \times SL(2) \times SL(n)$ automorphic forms, where, from (9.41) we see that the the abelian $GL(1)$ factor gives a power of the effective coupling.

A generic higher derivative term in Einstein frame in d dimensions may be written

$$l_d^{l-d} \int d^d x \sqrt{-g_e}\, \Phi_{E_{n+1}} \mathcal{O}, \tag{9.100}$$

where $\mathcal{O}$ is some polynomial in the curvature R, Cartan forms P or field strengths F. In the large volume limit $V_n \to \infty$ we have

$$\lim_{V_n \to \infty} l_{10}^n \int d^d x \sqrt{-g} V_n = \int d^{10} x \sqrt{-\hat{g}}, \tag{9.101}$$

where $\hat{}$ denotes a ten dimensional field. Examining the large volume limit of a generic higher derivative term in Einstein frame we see

$$\begin{aligned} \lim_{V_n \to \infty} l_d^{l-d} \int d^d x \sqrt{-g} \Phi_{E_{n+1}} \mathcal{O} &= \lim_{V_n \to \infty} l_{10}^{l-(10-n)} V_n^{-\frac{l-(10+n)}{8-n}} \int d^d x \sqrt{-g} V_n V_n^{-1} \Phi_{E_{n+1}} \mathcal{O} \\ &= l_{10}^{l-10} \int d^{10} x \sqrt{-\hat{g}} \lim_{V_n \to \infty} \left(V_n^{-\frac{l-2}{8-n}} \Phi_{E_{n+1}} \mathcal{O} \right). \end{aligned} \tag{9.102}$$

Note that in the last line we have made use of (9.101). Thus, demanding that the large volume limit of this generic higher derivative term exists from a string theory perspective means that the limit in the last line of (9.102) exists and that the resulting terms are ten dimensional higher derivative terms with a coefficient function that is a sum of $SL(2)$ automorphic forms (or zero). We may write the convergence condition as

$$\lim_{V_{n(B)} \to \infty} \left(V_n^{-\frac{l-2}{8-n}} \Phi_{E_{n+1}} \mathcal{O} \right) = \sum_j \sum_k a_k \hat{\Phi}^{(k)}_{SL(2)} \hat{\mathcal{O}}^j, \tag{9.103}$$

where k labels the $SL(2)$ automorphic forms that the E_{n+1} automorphic form splits into in the $V_{n(B)} \to \infty$ limit and a_k are constants that depend on the E_{n+1} automorphic form, while j labels the different $d = 10$ type IIB polynomials in the ten dimensional curvature $\hat{R}$, Cartan form $\hat{P}$ and field strengths $\hat{F}$ that arise in the decompactification of the d dimensional polynomial in the curvature R, Cartan forms P and field strengths F. Any higher derivative term in d dimensions that converges to a higher derivative term that is not compatible with type IIB string theory in $d = 10$ dimensions, must be rejected as a possible higher derivative term in d dimensions.

9.4.2 Decompactification of a Single Dimension Limit

Type II string theory in $d = 10 - n$ dimensions exhibits an $E_{n+1}(\mathbb{Z})$ symmetry. So, in the single dimension decompactification limit $\frac{r_{d+1}}{l_d} \to \infty$ an arbitrary higher derivative term in $d = 10 - n$ dimensions should have an expansion in powers of the dimensionless ratio $\frac{r_{d+1}}{l_d}$ in $d+1$ dimensions, whose coefficient functions are $E_n(\mathbb{Z})$ automorphic forms. The ratio of the radius in the compact $d+1$ direction to the d dimensional Planck length l_d, in $d = 10 - n$ dimensions, may be expressed in terms of the E_{n+1} Chevalley fields, as shown in equation (9.43), one finds

$$\frac{r_{d+1}}{l_d} = \frac{l_{10}}{l_d} \frac{r_{d+1}}{l_{10}} = e^{\frac{n}{8-n}\beta\rho} e^{\beta\rho + h^{d+1}{}_{d+1}} = e^{\dot{\varphi}_{d+1}}, \tag{9.104}$$

So the decompactification of a single compact dimension limit $\frac{r_{d+1}}{l_d} \to \infty$ is equivalent to $\dot{\varphi}_{d+1} \to \infty$. Taking $\dot{\varphi}_{d+1} \to \infty$ corresponds to deleting node 1 in the E_{n+1} Dynkin diagram, as displayed in figure 19 This decomposes the E_{n+1} algebra with respect to a $GL(1) \times E_n$ subalgebra. Taking

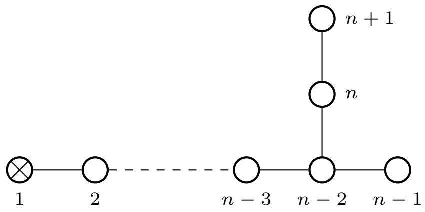

Figure 19: Dynkin diagram for E_{n+1} with node 1 deleted

the large volume limit $\frac{r_{d+1}}{l_d} \to \infty$ then splits an E_n automorphic form into a sum of $GL(1) \times E_n$ automorphic forms, where, from (9.104) we see that the the abelian $GL(1)$ factor gives a power of the dimensionless ratio $\frac{r_{d+1}}{l_d}$.

A generic higher derivative term in Einstein frame in d dimensions may be written

$$l_d^{l-d} \int d^d x \sqrt{-g_e} \Phi_{E_{n+1}} \mathcal{O}, \tag{9.105}$$

where $\mathcal{O}$ is some polynomial in the curvature R, Cartan forms P or degree m field strengths F_m. In the decompactification of a single compact dimension limit $\frac{r_{d+1}}{l_{d+1}} \to \infty$ we have

$$\lim_{\frac{r_{d+1}}{l_{d+1}} \to \infty} l_{d+1} \int d^d x \sqrt{-g} \frac{r_{d+1}}{l_{d+1}} = \int d^{d+1} x \sqrt{-\hat{g}}, \tag{9.106}$$

where $\hat{}$ denotes a $d+1$ dimensional field. We now examine the decompactification of a single dimension limit of a generic higher derivative term in Einstein frame. We will be interested in expressing the limit in terms of $d+1$ dimensional quantities, using (9.14) one may write

$$e^{\dot{\varphi}_{d+1}} = \frac{r_{d+1}}{l_d} = l_{d+1}^{-\frac{(9-n)}{8-n}} r_{d+1}^{\frac{9-n}{8-n}}, \tag{9.107}$$

we then take the limit $\frac{r_{d+1}}{l_{d+1}} \to \infty$. One finds

$$\begin{aligned} \lim_{\frac{r_{d+1}}{l_{d+1}} \to \infty} l_d^{l-d} \int d^d x \sqrt{-g} \Phi_{E_{n+1}} \mathcal{O} &= \lim_{\frac{r_{d+1}}{l_{d+1}} \to \infty} \int d^d x \sqrt{-g} l_{d+1}^{\frac{(d-1)(l-d)}{d-2}} r_{d+1}^{-\frac{l-d}{d-2}} \Phi_{E_{n+1}} \mathcal{O} \\ &= \int d^{d+1} x \sqrt{-\hat{g}} \lim_{\frac{r_{d+1}}{l_{d+1}} \to \infty} l_{d+1}^{\frac{(d-1)(l-d)}{d-2}} r_{d+1}^{\frac{2-l}{d-2}} \sum_k a_k \Phi^k_{E_{n+1}} \vartheta. \end{aligned} \tag{9.108}$$

Thus, demanding that the decompactification of a single dimension limit of this generic higher derivative term exists from a string theory perspective means that the limit in the last line of (9.108) exists and is an $E_n(\mathbb{Z})$ automorphic form in d dimensions that is compatible with type II string theory in $d+1$ dimensions. We may write the convergence condition as

$$\lim_{\frac{r_{d+1}}{l_{d+1}}\to\infty} l_{d+1}^{\frac{(d-1)(l-d)}{d-2}} r_{d+1}^{\frac{2-l}{d-2}} \sum_k a_k \Phi^k_{E_{n+1}} \mathcal{O} = \sum_j \sum_k a_k \Phi^k_{E_n} \hat{\mathcal{O}}^j, \tag{9.109}$$

where k labels the E_n automorphic forms that the E_{n+1} automorphic form splits into in the $\frac{r_{d+1}}{l_{d+1}} \to \infty$ limit and a_k are constants that depend on the E_{n+1} automorphic form, while j labels the different $d+1$ dimensional polynomials in the $d+1$ dimensional curvature $\hat{R}$, Cartan form $\hat{P}$ and field strengths $\hat{F}$ that arise in the decompactification of the d dimensional polynomial in the curvature R, Cartan forms P and field strengths F.

A d dimensional higher derivative term that converges to a $d+1$ dimensional higher derivative term with coefficient function that is incompatible with type II string theory in $d+1$ dimensions can not be a valid d dimensional type II string theory higher derivative term. For instance, we will see that the unconstrained automorphic form constructed out of the **133** of E_7 does not give a valid perturbative expansion in $d=4$ dimensions, therefore any higher derivative term in $d=3$ dimensions that converges to a $d=4$ higher derivative term that possesses the unconstrained Eisenstein-like automorphic form constructed out of the **133** of E_7 as a coefficient function, in the decompactification of a single dimension limit, is not a valid higher derivative term in $d=3$ dimensions.

9.4.3 Perturbative Limit

String perturbation theory in $d=10-n$ dimensions is an expansion in powers of the effective coupling g_d. The perturbative theory exhibits an $SO(n,n)$ symmetry, so in the perturbative limit $g_d \to 0$ an E_{n+1} automorphic form should have an expansion in powers of the effective coupling in d dimensions whose coefficient functions are $SO(n,n)$ automorphic forms. The coupling g_d in $d=10-n$ dimensions may be expressed in terms of the E_{11} Chevalley fields, one finds

$$g_d = e^{-2\left(\frac{8-n}{8}\right)\dot{\varphi}_{10}}, \tag{9.110}$$

where, again, we have made use of (9.40). So the perturbative limit $g_d \to 0$ in d dimensions is equivalent to $\dot{\varphi}_{10} \to \infty$. Taking $\dot{\varphi}_{10} \to \infty$ corresponds to deleting node $n+1$ in the E_{n+1} Dynkin diagram, as shown in figure (20) This splits the E_{n+1} algebra into a $GL(1) \times SO(n,n)$ subalgebra. Therefore, taking the perturbative limit $g_d \to 0$ splits an E_{n+1} automorphic form into a sum of $GL(1) \times SO(n,n)$ automorphic forms, where, from (9.110) we see that the abelian $GL(1)$ factor gives a power of the effective coupling g_d.

We require that the perturbative terms are consistent with a perturbative expansion in g_d. In string frame this translates to each term having coefficient g_d^{2g-2} where g is the genus. String frame in d dimensions is related to Einstein frame by $g_{E\mu\nu} = g_d^{-\frac{4}{d-2}} g_{s\mu\nu}$. Upon rescaling to string frame

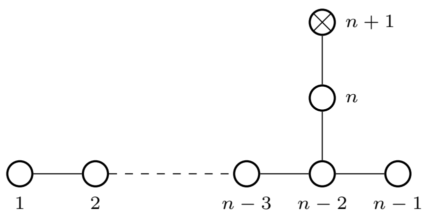

Figure 20: Dynkin diagram for E_{n+1} with node $n+1$ deleted

we find

$$\int d^d x \sqrt{-g_S} g_d^{\frac{4\Delta-2d}{d-2}} \Phi_{E_{n+1}} \mathcal{O}_S, \tag{9.111}$$

where $\mathcal{O}$ is some polynomial in the d dimensional curvature R, Cartan forms P or field strengths F, the subscript S denotes string frame quantities and Δ is the number of space time metrics that transform as a contravariant tensor minus the number of space time metrics that transform as a covariant tensor in $\mathcal{O}_S$. Examining the perturbative limit of a generic higher derivative term in string frame we see

$$\lim_{g_d\to 0} \int d^d x \sqrt{-g_S} g_d^{\frac{4\Delta-2d}{d-2}} \Phi_{E_{n+1}} \mathcal{O}_S = \lim_{g_d\to 0} \int d^d x \sqrt{-g_S} g_d^{\frac{4\Delta-2d}{d-2}} \sum_k \left(a_k g_d^{b_k} \Phi^{(k)}_{SO(n,n)} \right) \mathcal{O}_S, \tag{9.112}$$

where k labels the $SO(n,n)$ automorphic forms that the E_{n+1} automorphic form splits into in the $g_d \to 0$ limit and a_k, b_k are constants that depend on the E_{n+1} automorphic form. Thus, demanding that the perturbative limit of this generic higher derivative term exists from a string theory perspective means that the limit in the last line of (9.112) agrees with a perturbative expansion in g_d, this means that each term must be multiplied by a factor of the form g_d^{-2+2n}, where n is either zero or a positive integer. We may write this condition as

$$\lim_{g_d\to 0} g_d^{\frac{4\Delta-2d}{d-2}} \Phi_{E_{n+1}} = \sum_k a_k g_d^{-2+2n_k} \Phi^{(k)}_{SO(n,n)}, \tag{9.113}$$

where n_k, the genus of the d dimensional perturbative contribution to the higher derivative term, is a non-negative integer. Any suitable automorphic form should give an expansion in the effective coupling g_d that agrees with type II string theory in d dimensions with coefficient functions that are $SO(n,n)$ automorphic forms.

9.4.4 M-theory Limit

Type II string theory in d dimensions may be decompactified to eleven dimensional supergravity on an $(n+1)$ torus by taking the limit $V_m \to \infty$. In this limit the $SL(n+1)$ symmetry of the M-theory torus is manifest, so an arbitrary d dimensional higher derivative term in $d = 10 - n$ dimensions should permit an expansion in powers of the $(n+1)$ torus V_m whose coefficient functions are $SL(n+1,\mathbb{Z})$ automorphic forms. One may express the volume of the M-theory torus V_m in terms of the E_{11} Chevalley basis fields. Making use of (9.54), we see that,

$$V_m = e^{\frac{8-n}{3}\dot{\varphi}_{11}}, \tag{9.114}$$

so the M-theory limit $V_m \to \infty$ is equivalent to $\dot{\varphi}_{11} \to \infty$. Taking $\dot{\varphi}_{11} \to \infty$ corresponds to deleting node $n-1$ in the E_{n+1} Dynkin diagram, as displayed in figure 21.

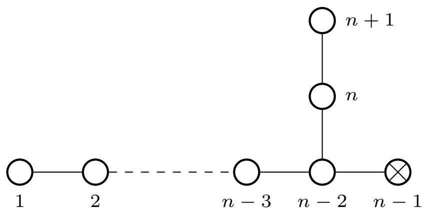

Figure 21: Dynkin diagram for E_{n+1} with node $n-1$ deleted

This splits the E_{n+1} algebra into a $GL(1) \times SL(n+1)$ subalgebra. Taking the large volume limit of the M-theory torus $V_m \to \infty$ then splits an E_{n+1} automorphic form into a sum of $GL(1) \times SL(n+1)$ automorphic forms, where, from (9.114) we see that the abelian $GL(1)$ factor gives a power of the volume of the M-theory torus V_m.

A generic higher derivative term in Einstein frame in d dimensions may be written

$$l_d^{l-d} \int d^d x \sqrt{-g_e}\, \Phi_{E_{n+1}} \mathcal{O}, \tag{9.115}$$

where $\mathcal{O}$ is some polynomial in the curvature R, Cartan forms P or degree k field strengths F_k and l counts the number of derivatives in $\mathcal{O}$. In the M-theory large volume limit we have

$$\lim_{V_m \to \infty} l_{11}^{n+1} \int d^d x \sqrt{-g} V_m = \int d^{11} x \sqrt{-\hat{g}}, \tag{9.116}$$

where, in this case, a hat $\hat{}$ denotes an eleven dimensional field. Using (9.24) we may write the d dimensional Planck length as a function of the eleven dimensional Planck length l_{11} and the M-theory torus V_m, then examining the M-theory limit $V_m \to \infty$ of a generic higher derivative term in Einstein frame in $d = 10 - n$ dimensions we see

$$\begin{aligned} \lim_{V_m \to \infty} l_{11}^{l-d} V_m^{-\frac{l-d}{8-n}} \int d^d x \sqrt{-g}\, \Phi_{E_{n+1}} \mathcal{O} &= \lim_{V_m \to \infty} l_{11}^{l-d} \int d^d x \sqrt{-g} V_m V_m^{-\frac{l-2}{8-n}} \sum_k \left(c_k V_m^{d_k} \Phi^{(k)}_{SL(n+1)} \right) \mathcal{O} \\ &= \int d^{11} x \sqrt{-\hat{g}} \lim_{V_m \to \infty} V_m^{-\frac{l-2}{8-n}} \sum_k \left(c_k V_m^{d_k} \Phi^{(k)}_{SL(n+1)} \right) \mathcal{O}, \end{aligned} \tag{9.117}$$

where c_k, d_k are constants that depend on the E_{n+1} automorphic form $\Phi_{E_{n+1}}$. Note that in the last line we have made use of (9.116). The M-theory limit of this generic higher derivative term is valid if the limit in the last line of (9.117) exists and is a sum of $SL(n+1, \mathbb{Z})$ automorphic forms which agree with the scattering amplitudes of eleven dimensional quantum supergravity compactified on V_m and vanish in the $V_m \to \infty$ limit if the $SL(n+1,\mathbb{Z})$ automorphic form is not trivial, since the higher derivative terms in the eleven dimensional supergravity theory cannot depend on the moduli of the torus. We may write the convergence requirement as

$$\lim_{V_m \to \infty} V_m^{-\frac{l-2}{8-n}} \sum_k \left(c_k V_m^{d_k} \Phi^{(k)}_{SL(n+1)} \right) \mathcal{O} = \sum_j a_j \hat{\mathcal{O}}^j. \tag{9.118}$$

where $a_j \in \mathbb{R}$ are constants and $\hat{\mathcal{O}}$ is a polynomial in the eleven dimensional curvature $\hat{R}$ and the four-form field strength $\hat{F}$, while j labels the different eleven dimensional polynomials in the eleven dimensional curvature $\hat{R}$ and field strengths $\hat{F}$ that arise in taking the M-theory limit of the d dimensional polynomial in the curvature R, Cartan forms S and field strengths F.

9.4.5 IIA Volume Limit

Type IIA string theory in $d = 10$ dimensions possesses a global $GL(1,\mathbb{R})$ symmetry. In this case the scalar sector that parameterises the coset associated with the global $GL(1,\mathbb{R})$ symmetry is trivial. However, in the large volume limit $V_{n(A)} \to \infty$, one still requires that the higher derivative terms in the effective action of the type IIA theory in $d = 10$ dimensions in string frame be multiplied by a factor of $e^{(-2+2g)\phi}$ where g is the genus of the $d = 10$ type IIA perturbative contribution.

The volume of the n torus in the IIA theory may be expressed in terms of the E_{11} Chevalley fields, one finds

$$V_n = \frac{r_{10} r_9 \ldots r_{d+1}}{l_{10}^n} = e^{\frac{8-n}{8}(\dot{\varphi}_{10} + 2\dot{\varphi}_{11})}, \tag{9.119}$$

where we have made use of equation (9.66). The volume of the IIA torus $V_{n(A)}$ depends on the two E_{n+1} Chevalley fields $\dot{\varphi}_{10}$ and $\dot{\varphi}_{11}$, since the IIA dilaton in $d = 10$ dimensions $\tilde{\sigma}$ is also a function of the these two E_{n+1} Chevalley fields, through $\tilde{\sigma} = -\frac{3}{2}\dot{\varphi}_{10} + \dot{\varphi}_{11}$, we must be careful to fix the linear combination $-\frac{3}{2}\dot{\varphi}_{10} + \dot{\varphi}_{11}$ of E_{n+1} Chevalley fields while taking the limit $(\dot{\varphi}_{10} + 2\dot{\varphi}_{11})$ to preserve the IIA coupling $g_{s(A)} = e^{\tilde{\sigma}}$ in the $d = 10$ type IIA theory. Taking $(\dot{\varphi}_{10} + 2\dot{\varphi}_{11}) \to \infty$ corresponds to deleting nodes $n-1$ and $n+1$ in the E_{n+1} Dynkin diagram, as displayed in figure (22). This splits the E_{n+1} algebra into a $GL(1) \times GL(1) \times SL(n)$ subalgebra. Taking the large

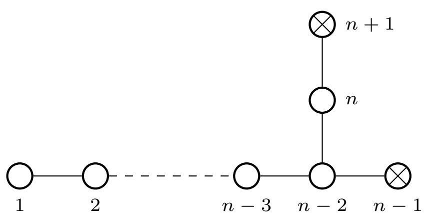

Figure 22: Dynkin diagram for E_{n+1} with nodes $n-1$ and $n+1$ deleted

volume limit $V_{n(A)} \to \infty$ splits an E_{n+1} automorphic form into a sum of $GL(1) \times GL(1) \times SL(n)$

automorphic forms, where, from (9.119) we see that the the abelian $GL(1)$ factors give powers of the volume of the IIA torus $V_{n(A)}$ and the IIA string coupling $g_{s(A)}$.

A generic higher derivative term in Einstein frame in d dimensions may be written

$$l_d^{l-d}\int d^dx\sqrt{-g_e}\Phi_{E_{n+1}}\mathcal{O}, \tag{9.120}$$

where $\mathcal{O}$ is some polynomial in the curvature R, Cartan forms P or field strengths F.

In the large volume limit $V_{n(A)}\to\infty$ we have

$$\lim_{V_{n(A)}\to\infty} l_{10(A)}^n\int d^dx\sqrt{-g}V_{n(A)} = \int d^{10}x\sqrt{-\hat{g}}, \tag{9.121}$$

where $\hat{}$ denotes a ten dimensional field. Examining the large volume limit of the IIA torus of a generic higher derivative term in Einstein frame we see

$$\begin{aligned}\lim_{V_{n(A)}\to\infty} l_d^{l-d}\int d^dx\sqrt{-g}\Phi_{E_{n+1}}\mathcal{O} &= \lim_{V_n\to\infty} l_{10}^{l-(10-n)}V_n^{-\frac{l-(10+n)}{8-n}}\int d^dx\sqrt{-g}V_nV_n^{-1}\Phi_{E_{n+1}}\mathcal{O}\\ &= l_{10}^{l-10}\int d^{10}x\sqrt{-\hat{g}}\lim_{V_n\to\infty}\left(V_n^{-\frac{l-2}{8-n}}\Phi_{E_{n+1}}\mathcal{O}\right),\end{aligned} \tag{9.122}$$

where we have made use of (9.121). The IIA limit of this generic higher derivative term is valid if the limit in the last line of (9.122) exists and is a sum of $SL(n,\mathbb{Z})$ automorphic forms that vanish in the $V_m\to\infty$ limit if the $SL(n,\mathbb{Z})$ automorphic form multiplying a decompactified higher derivative term is not trivial, since the higher derivative terms in type IIA string theory in $d=10$ cannot depend on the moduli of the torus. We may write the convergence requirement as

$$\lim_{V_{n(A)}\to\infty} V_{n(A)}^{-\frac{l-2}{8-n}}\Phi_{E_{n+1}}\mathcal{O} = \sum_k\sum_j a_k g_{s(A)}^{b_k}\hat{\mathcal{O}}_j, \tag{9.123}$$

where k labels the splitting of the E_{n+1} automorphic form into trivial $SL(n)$ automorphic forms with different powers of the type IIA string coupling $g_{s(A)}$. The constants a_k and b_k depend on $\Phi_{E_{n+1}}$ and $\hat{\mathcal{O}}$ is a polynomial in the type IIA $d=10$ curvature $\hat{R}$, Cartan forms $\hat{S}$ and field strengths $\hat{F}$, while j labels the different type IIA $d=10$ dimensional polynomials in the type IIA $d=10$ dimensional curvature $\hat{R}$, Cartan form $\hat{S}$ and field strengths $\hat{F}$ that arise in the decompactification of the d dimensional polynomial in the curvature R, Cartan forms S and field strengths F.

A further constraint is obtained by transforming the decompactified $d=10$ higher derivative terms to string frame. In string frame the $d=10$ type IIA string theory effective action must agree with a perturbative expansion in the type IIA string coupling $g_{s(A)}$, therefore one must find a coefficient of the form $g_{s(A)}^{-2+2g}$ for each decompactified higher derivative term, where g is the genus of the perturbative contribution to the decompactified higher derivative term. The Einstein frame metric g_E is related to the string frame metric by $g_E = e^{-\frac{1}{2}\tilde{\sigma}}g_S$, upon rescaling to string frame one finds that the requirement that the decompactified type IIA higher derivative terms on the right hand side of equation (9.123) have the correct dependence on the type IIA string coupling $g_{s(A)}$ to agree with a perturbative expansion in $g_{s(A)}$ becomes

$$\sum_k \sum_j a_k g_{s(A)}^{b_k} \hat{\mathcal{O}}^j = g_{s(A)}^{-\frac{5}{2}} \sum_k \sum_j a_k g_{s(A)}^{b_k} g_{s(A)}^{\frac{1}{2}\hat{\Delta}_j} \hat{\mathcal{O}}_{Sj}, \tag{9.124}$$

where $\hat{\Delta}_j$ is the number $d = 10$ metrics that transform as a contravariant tensor minus the number of space time metrics that transform as a covariant tensor in $\hat{\mathcal{O}}_{Sj}$ and the factor of $g_{s(A)}^{-\frac{5}{2}}$ arises from transforming the factor $\sqrt{-\hat{g}}$ to string frame. Therefore from (9.124) we find that for a decompactified type IIA higher derivative term to agree with a perturbative expansion in the type IIA coupling $g_{s(A)}$ one requires,

$$-\frac{5}{2} + b_k + \frac{1}{2}\hat{\Delta}_j = -2 + 2g_{kj}, \tag{9.125}$$

where g_{kj} is the genus of the perturbative contribution for the polynomial in the type IIA curvature R, Cartan forms P and field strengths F labelled by j.

9.5 Limits of the Unconstrained Eisenstein-like Automorphic Form

We will now evaluate the unconstrained automorphic form given in [40] in the five limits described in the previous section. Four of these limits, the d dimensional perturbative limit $g_d \to 0$, the large volume limit of the M-theory torus $V_m \to \infty$, the decompactification of a single dimension limit $\frac{r_{d+1}}{l_d} \to \infty$ and the large volume limit of the IIB torus $V_{n(B)} \to \infty$ involve the deletion of a single node of the E_{n+1} Dynkin diagram. One may write down a general formulation of the limit of the unconstrained automorphic form, in the case where taking a limit corresponds to deleting a single node of the E_{n+1} Dynkin diagram, and then examine each of these limits individually. However, the fifth limit we consider, the large volume limit of the type IIA torus $V_{n(A)} \to \infty$, involves deleting two nodes of the E_{n+1} Dynkin diagram. Although the techniques used in evaluating the unconstrained automorphic form in the $V_{n(A)} \to \infty$ limit are similar to those used in the evaluation of the other four limits, the deletion of an additional node slightly increases the complexity of the calculation, so we will first examine the general formulation of the limits of the unconstrained automorphic form involving the deletion of a single node of the E_{n+1} Dynkin diagram before evaluating the unconstrained automorphic form in the $V_{n(A)} \to \infty$ limit.

9.5.1 Evaluation of the Unconstrained Automorphic Form in a Single Deletion Limit

Generic higher derivative corrections in the effective action of Type II string theory, compactified on an n-torus to $d = 10 - n$ dimensions, are polynomials in the curvature R, Cartan forms S

and degree k field strengths F_k multiplied by an automorphic form $\Phi_{E_{n+1}}$ transforming under the E_{n+1} U-duality group. As described in chapter eight, one may construct an E_{n+1} automorphic form $\Phi_{E_{n+1}}$ from the function $|\varphi\rangle$, which is defined by,

$$|\varphi\rangle = L(g^{-1})|\psi\rangle \tag{9.126}$$

where $L(g^{-1})$ is a representation of the coset element $g \in E_{n+1}/H$, H being the maximal compact sub-group of E_{n+1} and $|\psi\rangle$ is a linear representation of $E_{n+1}(\mathbb{Z})$. Using the Iwasawa decomposition and fixing the local group element $h \in H$ to be the identity, we may write

$$L(g^{-1}) = e^{\frac{1}{\sqrt{2}}\vec{\phi}.\vec{H}} e^{-\sum_{\vec{\alpha}>0} \chi_{\vec{\alpha}} E_{\vec{\alpha}}}, \tag{9.127}$$

where $\vec{H}$ are the generators in the Cartan sub-algebra of E_{n+1}, in Weyl basis, and $E_{\vec{\alpha}}$ are the positive root generators, while $\chi_{\vec{\alpha}}$ are the axions and $\vec{\phi}$ is a vector whose components are linear combinations of the physical fields of type IIA/B string theory or M-theory, namely, the type IIA/B dilaton, the n-torus volume modulus ρ and the remaining n or $n-1$ moduli $\underline{\phi}$. Instead of writing the coset element $g \in E_{n+1}/K$ in terms of the type IIA, type IIB or M-theory physical fields we may write it as a function of the E_{n+1} Chevalley fields $\dot{\varphi}_i$, $i = d+1, ..., n+1$ parameterising the E_{n+1} symmetry. The fields in the E_{n+1} part of the E_{11} group element $\dot{\varphi}_i$ are equal to those in the physical field parameterisation of the group element $L\left(g^{-1}\right)$ used to construct the automorphic form, up to a numerical factor. We find, through comparing the normalisations of the fields associated with the Cartan subalgebra in the E_{n+1} part of the E_{11} group element $e^{\vec{\varphi}.\vec{H}}$ and those in the automorphic form group element $e^{-\frac{1}{\sqrt{2}}\vec{\phi}.\vec{H}}$ that $\phi_i = -\sqrt{2}\tilde{\varphi}_i$, where $\tilde{\varphi}_i$ are the E_{n+1} Chevalley fields in Weyl basis. So the coset element $g \in E_{n+1}/K$ as a function of the E_{n+1} fields $\tilde{\varphi}_i$ is

$$L(g^{-1}) = e^{-\vec{\varphi}.\vec{H}} e^{-\sum_{\vec{\alpha}>0} \chi_{\vec{\alpha}} E_{\vec{\alpha}}}, \tag{9.128}$$

where $\vec{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, ..., \tilde{\varphi}_{n+1})$. In Weyl basis, where the commutator of the Cartan subalgebra elements H_i with the positive root generators is $[H_i, E_{\vec{\alpha}}] = \vec{\alpha}_i E_{\vec{\alpha}}$, the action of the Cartan subalgebra $\vec{H}$ on $|\psi_{\vec{\Lambda}_k}\rangle$ is

$$\begin{aligned} |\varphi_{\Lambda_k}\rangle =& e^{-\vec{\varphi}.\vec{H}}|\psi_{\Lambda_k}\rangle \\ =& e^{-\vec{\varphi}.[\vec{\Lambda}_k]}|\psi_{\Lambda_k}\rangle. \end{aligned} \tag{9.129}$$

where $[\vec{\Lambda}_k]$ is the set of weights in the representation of E_{n+1} with highest weight $\vec{\Lambda}_k$. The Weyl basis of E_{n+1} fields $\tilde{\varphi}_i$ are related to the E_{n+1} Chevalley basis fields $\dot{\varphi}_a$ by $\tilde{\varphi}_i = \dot{\varphi}_a \alpha_i^a$, where α_i^a is the i'th component of the a'th simple root. We will denote the automorphic form that is a function of the above non-linearly realised $\vec{\Lambda_k}$ representation of E_{n+1} by $\Phi_{E_{n+1}}(|\varphi_{\Lambda_k}\rangle)$. An invariant $\Phi_{E_{n+1}}(|\varphi_{\lambda_k}\rangle)$ automorphic form is then given by

$$\Phi_{E_{n+1}}(|\varphi_{\lambda_k}\rangle) = \sum_{\Lambda} \frac{1}{(\langle\varphi_{D\tau}|\varphi\rangle)^s}, \tag{9.130}$$

where the sum is over all states $m_i \in \mathbb{Z}$, $i = 1, ..., N$ in the lattice Λ and $\langle\varphi_{D\tau}|$ is the twisted Cartan involution of $|\varphi\rangle$. For more details on this construction of an automorphic form see [40].

We will now examine the behaviour of the automorphic form $\Phi_{E_{n+1}}(|\varphi_{\lambda_k}\rangle)$ in these limits. As discussed earlier, taking the limit in some physical parameter in type IIA/B string theory and M-theory, in d dimensions corresponds to deleting a node in the E_{n+1} Dynkin diagram. Upon deleting node m, the E_{n+1} algebra, that consists of the Cartan subalgebra H_i, $i = 1, .., n+1$, and the positive root generators $E_{\vec{\alpha}}$ splits into a subalgebra generating a $GL(1) \times A$ subgroup. The E_{n+1} Cartan subalgebra element H_m is the generator of the abelian $GL(1)$ factor in the $GL(1) \times A$ subgroup, while the remaining E_{n+1} Cartan subalgebra generators H_i, $i \neq m$ and the positive root generators in $\left\{E_{\vec{\alpha}} : \vec{\alpha}.\vec{\Lambda}_m = 0\right\}$ generate the A subgroup. For simplicity we will set the generators $E_{\vec{\alpha}}$ corresponding to the positive roots $\vec{\alpha}$ satisfying $\vec{\alpha}.\Lambda_m \neq 0$ to zero, since one may show that the fields that parameterise these generators are suppressed in the limits we are considering. In general, the simple roots $\vec{\alpha}$ of E_{n+1} decompose as

$$\begin{aligned} \vec{\alpha}_i &= (0, \underline{\alpha}_i), \quad i \neq m, \\ \vec{\alpha}_m &= \left(x, -\underline{\lambda}_j\right), \end{aligned} \tag{9.131}$$

where $\underline{\alpha}_k$ and $\underline{\lambda}_k$ denote the simple roots and fundamental weights (respectively) of the subalgebra generating A, while x is fixed by requiring $\vec{\alpha}_m.\vec{\alpha}_m = 2$ and the fundamental weight $\underline{\lambda}_j$ depends on the deleted node and n. The corresponding fundamental weights are $\vec{\Lambda}^i = \left(\frac{\underline{\lambda}_i.\underline{\lambda}_j}{x}, \underline{\lambda}_i\right)$, $i \neq m$ and $\vec{\Lambda}^m = \left(\frac{1}{x}, \underline{0}\right)$. Under this decomposition any highest weight representation of the group element $g \in E_{n+1}/H$, in Weyl basis, becomes

$$\begin{aligned} L\left(g^{-1}\right) &= e^{-\tilde{\varphi}_1 H_1} e^{-\underline{\varphi}.\underline{H}} e^{-\sum_{\underline{a}>0} \underline{\chi}_{\underline{\alpha}} E_{\underline{\alpha}}} \\ &= e^{-\tilde{\varphi}_1 H_1} g_A^{-1}. \end{aligned} \tag{9.132}$$

In this expression $\tilde{\varphi}_1$ is the first component of the Weyl basis of E_{n+1} fields which are related to the E_{n+1} Chevalley basis fields by $\tilde{\varphi}_i = \dot{\varphi}_a \alpha_i^a$, where α_i^a is the i'th component of the a'th simple root and $\dot{\varphi}_a$ is the a'th Chevalley basis field. One finds $\tilde{\varphi}_1 = x\dot{\varphi}_m$, so the highest weight representation of the group element $g \in E_{n+1}/K$ becomes

$$L\left(g^{-1}\right) = e^{-x\dot{\varphi}_m H_1} L\left(g_A^{-1}\right). \tag{9.133}$$

The non-linearly realised lattice state $|\varphi\rangle$ is then

$$\begin{aligned}|\varphi\rangle &= \sum_i m_i e^{-x\dot{\varphi}_m H_1} e^{-\underline{\tilde{\varphi}}.\underline{\tilde{H}}} e^{-\sum_{\tilde{\alpha}>0}\chi_{\tilde{\alpha}}E_{\tilde{\alpha}}}|\mu_i\rangle \\ &= \sum_i m_i e^{-x\dot{\varphi}_m H_1} L\left(g_A^{-1}\right)|\mu_i\rangle,\end{aligned} \tag{9.134}$$

In the usual way, we find $u = \langle\varphi_{D\tau}|\varphi\rangle$ to be

$$\begin{aligned}u &= \sum_{i,j}\langle\vec{\mu}^j|m_j e^{-\sum_{\tilde{\alpha}>0}E_{-\tilde{\alpha}}\chi_{\tilde{\alpha}}} e^{-2\vec{\varphi}.\vec{H}} e^{-\sum_{\tilde{\alpha}>0}E_{\tilde{\alpha}}\chi_{\tilde{\alpha}}} m_i|\mu^i\rangle \\ &= \sum_k\sum_{i,j} m_i m_j\langle\vec{\mu}^j|L_A\left(\left(g_A^{-1}\right)^{\#}\right)e^{-2x\dot{\varphi}_m.H_1}L_A\left(g_A^{-1}\right)|\mu^i\rangle \\ &= \sum_{k\geq 0}\sum_l e^{-2\dot{\varphi}_m\left(b-kx^2\right)}\sum_{i,j} m_i m_j\langle\mu_j|L_A\left(\left(g^{-1}_{A_{(k,l)}}\right)^{\#}\right)L_A\left(g^{-1}_{A_{(k,l)}}\right)|\mu^i\rangle,\end{aligned} \tag{9.135}$$

where L_A is a representation of the subgroup A, b is a constant that depends on the highest weight representation of E_{n+1}, $k\in\mathbb{Z}$ is the level with respect to the $\vec{\alpha}_m$ simple root, l labels the representations of A at level k and $g^{-1}_{A_{(k,l)}(n,n)}$ is the A coset element acting on a subset of the E_{n+1} lattice states. The splitting of u into a sum over levels k is a consequence of the matrix element $\langle\mu_j|L\left(\left(g^{-1}_{A_{(k,l)}}\right)^{\#}\right)L\left(g^{-1}_{A_{(k,l)}}\right)|\mu^i\rangle$ vanishing between states $|\mu^i\rangle$ and $|\mu^j\rangle$ with different H_1 eigenvalues. The automorphic form constructed from φ is given by

$$\begin{aligned}\Phi_{E_{n+1}} &= \sum_\Lambda\frac{1}{\left(\langle\varphi|\varphi\rangle\right)^s} \\ &= \sum_\Lambda\frac{1}{\left(\left(\sum_{k\geq 0}\sum_l e^{-2\dot{\varphi}_m(b-kx^2)}\sum_{i,j} m_i m_j\langle\mu_j|L_A\left(\left(g^{-1}_{A_{(k,l)}}\right)^{\#}\right)L_A\left(g^{-1}_{A_{(k,l)}}\right)|\mu^i\rangle\right)\right)^s} \\ &= \frac{\pi^s}{\Gamma(s)}\sum_\Lambda\int_0^\infty\frac{dt}{t^{1+s}}e^{-\frac{\pi}{t}\left(\sum_{k\geq 0}\sum_l e^{-2\dot{\varphi}_m\left(b-kx^2\right)}\sum_{i,j} m_i m_j\langle\mu_j|L_A\left(\left(g^{-1}_{A_{(k,l)}}\right)^{\#}\right)L_A\left(g^{-1}_{A_{(k,l)}}\right)|\mu^i\rangle\right)}.\end{aligned} \tag{9.136}$$

Letting c be the number of representations of A contained in the highest weight representation of E_{n+1}. We then split the sum over Λ into c sums, defining $\gamma_{(q,r)}$ to be all m_i acted on by the group element $g^{-1}_{A_{(q,r)}}$ of the representation labelled by the level q and representation label r (at level q), except m_i all equal to zero, and all m_j acted on by representations $A_{(p,w)}$ satisfying $p<q$. One finds that

$$\sum_\Lambda = \sum_{q,r}\sum_{\gamma_{(q,r)}}, \tag{9.137}$$

where $\sum_{q,r}$ is a sum over all levels q and representations r at level q. So, we may write

$$\Phi_{E_{n+1}} = \frac{\pi^s}{\Gamma(s)} \sum_{q,r} \sum_{\gamma_{q,r}} \int_0^\infty \frac{dt}{t^{1+s}} e^{-\frac{\pi}{t}\left(\sum_{k\geq 0}\sum_l e^{-2\dot{\varphi}_m\left(b-kx^2\right)} \sum_{i,j} m_i m_j \langle\mu_j| \left(g^{-1}_{A_{(k,l)}}\right)^{\#} g^{-1}_{A_{(k,l)}} |\mu^i\rangle\right)}. \tag{9.138}$$

Now, in each sum over $\gamma_{q,r}$ we perform a Poisson resummation over all m_i acted on by the group elements $g^{-1}_{A(p,w)}$ of the representations labelled by the level p and representation labels w satisfying $p<q$. If we let M_k be equal to the number of states in the highest weight representation of E_{n+1} at level k we find, after Poisson resumming,

$$\begin{aligned}\Phi_{E_{n+1}} = \frac{\pi^s}{\Gamma(s)} & \sum_{q,r} \sum_{\hat{\gamma}_{q,r}} \sum_{\Lambda_{q,r}} \prod_{y<q,z} det(g_{A_{(y,z)}}) e^{\sum_{p<q} M_p\left(b-px^2\right)\dot{\varphi}_m} \int_0^\infty \frac{dt}{t^{1+s-\frac{1}{2}\sum_{p<q}M_p}} \\ & \times e^{-\pi t\left(\sum_{p<q}\sum_l e^{2\dot{\varphi}_m\left(b-px^2\right)} \sum_{i,j} \hat{m}_i \hat{m}_j \langle\mu_j| \left(g^{-1}_{A_{(p,l)}}\right)^{\#} g^{-1}_{A_{(p,l)}} |\mu^i\rangle\right)} \\ & \times e^{-\frac{\pi}{t} e^{-2\dot{\varphi}_m\left(b-qx^2\right)} \sum_{i,j} m_i m_j \langle\mu_j| \left(g^{-1}_{A_{(q,r)}}\right)^{\#} g^{-1}_{A_{(q,r)}} |\mu^i\rangle}.\end{aligned} \tag{9.139}$$

In this expression a $\hat{}$ denotes the Poisson resummed integers and $\gamma_{q,r}$ has been rewritten $\gamma_{q,r} = \sum_{\hat{\gamma}_{q,r}} \sum_{\Lambda_{q,r}}$, where $\sum_{\hat{\gamma}_{q,r}}$ is a sum over the Poisson resummed integers in $\gamma_{q,r}$ and $\sum_{\Lambda_{q,r}}$ is a sum over the lattice spanned by the m_i that are acted on by group element $g^{-1}_{A(q,r)}$ excluding the origin. Note that in most of the decompositions we consider the subalgebra group element $g_{A_{(y,z)}}$ is equal to one and in these cases $\prod_{y<q,z} det(g_{A_{(y,z)}}) = 1$. Changing variables in each sum by letting $t = ve^{-2\dot{\varphi}_m\left(b-qx^2\right)}$ gives,

$$\begin{aligned}\Phi_{E_{n+1}} = \frac{\pi^s}{\Gamma(s)} & \sum_{q,r} \sum_{\hat{\gamma}_{q,r}} \sum_{\Lambda_{q,r}} \prod_{y<q,z} det(g_{A_{(y,z)}}) e^{\sum_{p<q} M_p\left(b-px^2\right)\dot{\varphi}_m} e^{2\left(s-\frac{1}{2}\sum_{p<q}M_p\right)\left(b-qx^2\right)\dot{\varphi}_m} \\ & \times \int_0^\infty \frac{dv}{v^{1+s-\frac{1}{2}\sum_{p<q}M_p}} e^{-\pi v\left(\sum_{p<q}\sum_l e^{2\dot{\varphi}_m(q-p)x^2} \sum_{i,j} \hat{m}_i \hat{m}_j \langle\mu_j| \left(g^{-1}_{A_{(p,l)}}\right)^{\#} g^{-1}_{A_{(p,l)}} |\mu^i\rangle\right)} \\ & \times e^{-\frac{\pi}{v}\sum_{i,j} m_i m_j \langle\mu_j| \left(L\left(g^{-1}_{A_{(q,r)}}\right)^{\#}\right) L\left(g^{-1}_{A_{(q,r)}}\right) |\mu^i\rangle}.\end{aligned} \tag{9.140}$$

Since $L_A\left(\left(g^{-1}_{A_{(p,l)}}\right)^{\#}\right) L_A\left(g^{-1}_{A_{(p,l)}}\right)$ is a symmetric matrix and the states $|\mu^i\rangle$ are orthogonal, the coefficient of all terms with $\hat{m}_i \neq 0$ is negative and thus vanish in the limit $\dot{\varphi}_m \to \infty$. Therefore, the E_{n+1} automorphic form, in the $\dot{\varphi}_m \to \infty$ limit, can be written

$$\Phi^s_{E_{n+1}} = \frac{\pi^s}{\Gamma(s)} \sum_{q,r} \sum_{\Lambda_{q,r}} \prod_{y<q,z} det(g_{A_{(y,z)}}) e^{\left(2s\left(b-qx^2\right)+\sum_{p<q} M_p(q-p)x^2\right)\dot{\varphi}_m} \int_0^\infty \frac{dv}{v^{1+s-\frac{1}{2}\sum_{p<q} M_p}}$$
$$\times e^{-\frac{\pi}{v}\sum_{i,j} m_i m_j \langle \mu_j | L_A\left(\left(g^{-1}_{A_{(q,r)}}\right)^{\#}\right) L_A\left(g^{-1}_{A_{(q,r)}}\right)|\mu^i\rangle}$$
$$= \sum_{q,r} \pi^{\frac{1}{2}\sum_{p<q} M_p} \frac{\Gamma\left(s-\frac{1}{2}\sum_{p<q} M_p\right)}{\Gamma(s)} e^{\left(2s\left(b-qx^2\right)+\sum_{p<q} M_p(q-p)x^2\right)\dot{\varphi}_m} \Phi^{s-\frac{1}{2}\sum_{p<q} M_p}_{A_{(q,r)}}, \tag{9.141}$$

where $\Phi_{A_{(q,r)}}$ is the automorphic form constructed out of the r'th representation of A at level q in the decomposition of a highest weight representation of E_{n+1}. Writing the highest weight Λ_Φ of the representation used to construct the automorphic form $\Phi_{E_{n+1}}$ as a linear combination of the fundamental weights of E_{n+1}, $\Lambda_\Phi = \sum_{k=1}^{n+1} a_k \Lambda_k$, where the coefficients a_k are positive integers or zero, one finds,

$$b = x \sum_{k=1}^{n+1} a_k \Lambda_{k(1)}, \tag{9.142}$$

where $\Lambda_{k(1)}$ is the first component of the fundamental weight Λ_k.

Four of the limits we are interested in may now be evaluated by the deletion of the appropriate node.

9.5.2 The Perturbative Limit

The perturbative limit, $g_d \to 0$, is found by deleting node $n+1$ in the Dynkin diagram and decomposing a representation of E_{n+1} into representations of $GL(1) \times SO(n,n)$. Since $g_d = e^{-2\frac{(8-n)}{8}\dot{\varphi}_{10}}$ and $x^2 = \frac{8-n}{4}$ we may write $e^{\dot{\varphi}_{10}} = g_d^{-\frac{1}{x^2}}$. The perturbative limit of the automorphic form (9.141) is then

$$\Phi^s_{E_{n+1}} = \sum_{q,r} \pi^{\frac{1}{2}\sum_{p<q} M_p} \frac{\Gamma\left(s-\frac{1}{2}\sum_{p<q} M_p\right)}{\Gamma(s)} g_d^{-\left(2s\left(\frac{b}{x^2}-q\right)+\sum_{p<q} M_p(q-p)\right)} \Phi^{s-\frac{1}{2}\sum_{p<q} M_p}_{SO(n,n)_{(q,r)}}. \tag{9.143}$$

9.5.3 Large Volume Limit of the IIB Torus

The IIB volume limit, $V_{n(B)} \to \infty$, is found by deleting node n in the Dynkin diagram and decomposing a representation of E_{n+1} into representations of $GL(1) \times SL(2) \times SL(n)$. Since $V_{n(B)} = e^{\frac{(8-n)}{4}\dot{\varphi}_9}$ and $x^2 = \frac{8-n}{2n}$ we may write $e^{\dot{\varphi}_9} = V_{n(B)}^{\frac{2}{nx^2}}$. The IIB volume limit of the automorphic form (9.141) is then

$$\Phi^s_{E_{n+1}} = \sum_{q,r} \pi^{\frac{1}{2}\sum_{p<q} M_p} \frac{\Gamma\left(s-\frac{1}{2}\sum_{p<q} M_p\right)}{\Gamma(s)} V_{n(B)}^{\left(\frac{4}{n}s\left(\frac{b}{x^2}-q\right)+\frac{2}{n}\sum_{p<q} M_p(q-p)\right)} \Phi^{s-\frac{1}{2}\sum_{p<q} M_p}_{(SL(2)\times SL(n))_{(q,r)}}. \tag{9.144}$$

Writing the highest weight $\vec{\Lambda}_\Phi$ of the representation used to construct the automorphic form $\Phi_{E_{n+1}}$ as a linear combination of the fundamental weights of E_{n+1}, $\vec{\Lambda}_\Phi = \sum_{k=1}^{n+1} a_k \vec{\Lambda}_k$, where the coefficients a_k are positive integers or zero, one finds,

$$b = \sum_{k=1}^{n-2} \frac{8k}{n(8-n)} a_k + \frac{5n-8}{n(8-n)} a_n + \frac{3}{8-n} a_{n-1} + \frac{2}{8-n} a_{n+1} \tag{9.145}$$

9.5.4 Large Volume Limit of the M-theory Torus

The M-theory volume limit, $V_m \to \infty$, is found by deleting node $n-1$ in the Dynkin diagram and decomposing a representation of E_{n+1} into representations of $GL(1) \times SL(n+1)$. Since $V_m = e^{\frac{(8-n)}{3}\dot{\varphi}_{11}}$ and $x^2 = \frac{8-n}{n+1}$ we may write $e^{\dot{\varphi}_{11}} = V_m^{\frac{3}{(n+1)x^2}}$. The M-theory volume limit of the automorphic form (9.141) is then

$$\Phi^s_{E_{n+1}} = \sum_{q,r} \pi^{\frac{1}{2}\sum_{p<q} M_p} \frac{\Gamma\left(s - \frac{1}{2}\sum_{p<q} M_p\right)}{\Gamma(s)} V_m^{\frac{3}{n+1}\left(2s\left(\frac{b}{x^2}-q\right)+\sum_{p<q} M_p(q-p)\right)} \Phi^{s-\frac{1}{2}\sum_{p<q} M_p}_{(SL(n+1))_{(q,r)}}. \tag{9.146}$$

9.5.5 Decompactification of a Single Dimension Limit

The decompactification of a single dimension limit, $\frac{r_{d+1}}{l_d}$, is found by deleting node 1 in the Dynkin diagram and decomposing a representation of E_{n+1} into representations of $GL(1) \times E_n$. Since $\frac{r_{d+1}}{l_{d+1}} = e^{\left(\frac{8-n}{9-n}\right)\dot{\varphi}_{d+1}}$ and $x^2 = \frac{8-n}{9-n}$ we may write

$$e^{\dot{\varphi}_{d+1}} = \left(\frac{r_{d+1}}{l_{d+1}}\right)^{\frac{1}{x^2}}. \tag{9.147}$$

The decompactification of a single dimension limit of the automorphic form (9.141) is then

$$\Phi^s_{E_{n+1}} = \sum_{q,r} \prod_{y<q,z} det(g_{A_{(y,z)}}) \pi^{\frac{1}{2}\sum_{p<q} M_p} \frac{\Gamma\left(s - \frac{1}{2}\sum_{p<q} M_p\right)}{\Gamma(s)} \frac{r_{d+1}}{l_{d+1}}^{\left(2s\left(\frac{b}{x^2}-q\right)+\sum_{p<q} M_p(q-p)\right)} \Phi^{s-\frac{1}{2}\sum_{p<q} M_p}_{(E_n)_{(q,r)}}. \tag{9.148}$$

9.5.6 Evaluation of the Large Volume Limit of the IIA Torus

The large volume limit of the IIA torus $V_{n(A)} \to \infty$, involves the deletion of nodes 10 and 11 of the E_{11} Dynkin diagram. The E_{n+1} algebra made up of the Cartan subalgebra, H_i, $i = 1, 2, ..., n+1$ and the positive root generators $E_{\vec{\alpha}}$ splits into a subalgebra generating a $GL(1) \times GL(1) \times SL(n)$ subgroup. Two E_{n+1} Cartan subalgebra generators, H_1 and H_2, generate the abelian $GL(1)$ factors while the remaining E_{n+1} Cartan subalgebra generators H_i, $i = 3, ..., n+1$ and the E_{n+1} positive root generators lying in the set $\left\{E_{\vec{\alpha}} : \vec{\alpha}.\vec{\Lambda}_{n-1} = \vec{\alpha}.\vec{\Lambda}_{n+1} = 0\right\}$ generate the $SL(n)$ subgroup. Details of the decomposition of the E_{n+1} algebra with respect to the $GL(1) \times GL(1) \times SL(n)$ subalgebra found by deleting nodes 10 and 11 in the E_{11} Dynkin diagram are given in appendix $D.5$. The simple roots $\vec{\alpha}$ of E_{n+1} decompose as

$$\begin{aligned}
\vec{\alpha}_i &= (0,0,\underline{\alpha}_i)\,, \quad i=1,...,n-2,\\
\vec{\alpha}_{n-1} &= \left(x, -\frac{\lambda_{n-2}\cdot\lambda_{n-1}}{y}, -\underline{\lambda}_{n-1}\right),\\
\vec{\alpha}_n &= \left(0,0,\underline{\alpha}_{n-1}\right),\\
\vec{\alpha}_{n+1} &= \left(0,y,-\underline{\lambda}_{n-2}\right),
\end{aligned} \tag{9.149}$$

where $\underline{\lambda_k}$ denotes the simple roots and fundamental weights of the subalgebra generating the $SL(n)$ subgroup. One finds $x^2=\frac{8-n}{4}$ and $y^2=\frac{4}{n}$. The corresponding fundamental weights are

$$\vec{\Lambda}_i = \left(\frac{c_i}{x}, \frac{\lambda_{n-2}\cdot\lambda_i}{y}, \underline{\lambda}_i\right), \quad i=1,...,n-1, \quad \vec{\Lambda}_n = \left(\frac{1}{x},0,\underline{0}\right), \quad \vec{\Lambda}_{n+1} = \left(\frac{n-2}{4x}, \frac{1}{y}, \underline{0}\right), \tag{9.150}$$

where $c_i=\frac{i}{2}$, $i=1,\ldots,n-2$ and $c_{n-1}=\frac{n}{4}$. As $\tilde{\underline{\Lambda}}^2_{n-1}=\frac{n}{4}$ we find that $x^2=\frac{8-n}{4}$. Under this decomposition any highest weight representation of the group element $g\in E_{n+1}/K$, in Weyl basis, becomes

$$\begin{aligned}
g^{-1} &= e^{\frac{1}{\sqrt{2}}\tilde{\varphi}_1 H_1} e^{\frac{1}{\sqrt{2}}\tilde{\varphi}_2 H_2} e^{\frac{1}{\sqrt{2}}\underline{\tilde{\varphi}}\cdot\underline{H}} e^{-\sum_{\underline{\alpha}>0}\underline{\chi}_{\underline{\alpha}}E_{\underline{\alpha}}}\\
&= e^{\frac{1}{\sqrt{2}}(\tilde{\varphi}_1 H_1+\tilde{\varphi}_2 H_2)} g_A^{-1}.
\end{aligned} \tag{9.151}$$

In this expression $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are the first and second components, respectively, of the Weyl basis of E_{n+1} fields which are related to the E_{n+1} Chevalley basis fields by $\tilde{\varphi}_i=\dot{\varphi}_a\alpha_i^a$, where α_i^a is the i'th component of the a'th simple root and ϕ_a is the a'th Chevalley basis field and H_k are the Cartan subalgebra generators in Weyl basis that satisfy the commutation relations $[H_k,E_{\vec{\alpha}}]=\vec{\alpha}_k E_{\vec{\alpha}}$ with the positive root generators $E_{\vec{\alpha}}$. One finds $\tilde{\varphi}_1=x\dot{\varphi}_{10}$ and $\tilde{\varphi}_2=-\frac{\lambda_{n-2}\cdot\lambda_{n-1}}{y}\dot{\varphi}_{10}+y\dot{\varphi}_{11}$, so the highest weight representation of the group element $g\in E_{n+1}/K$ may be written

$$g^{-1} = e^{\frac{1}{\sqrt{2}}\left(x\dot{\varphi}_{10}H_1+\left(-\frac{\lambda_{n-2}\cdot\lambda_{n-1}}{y}\dot{\varphi}_{10}+y\dot{\varphi}_{11}\right)H_2\right)} g_{SL(n)}^{-1}. \tag{9.152}$$

The non-linearly realised lattice state $|\varphi\rangle$ is

$$|\varphi\rangle = \sum_i m_i e^{\frac{1}{\sqrt{2}}\left(x\dot{\varphi}_{10}H_1+\left(-\frac{\lambda_{n-2}\cdot\lambda_{n-1}}{y}\dot{\varphi}_{10}+y\dot{\varphi}_{11}\right)H_2\right)} g_{SL(n)}^{-1}. \tag{9.153}$$

One then finds

$$
\begin{aligned}
u &= \langle \varphi_{D\tau} | \varphi \rangle \\
&= \sum_{(k_1,k_2)\geq(0,0)} \sum_{l} e^{-2\left((b_1\dot{\varphi}_{10}+b_2\dot{\varphi}_{11})-k_{n-1}\left(-\underline{\lambda}_{n-2}\cdot\underline{\lambda}_{n-1}\dot{\varphi}_{10}+y^2\dot{\varphi}_{11}\right)\right)} \\
&\quad \times e^{2k_{n+1}\left(\left(x^2+\frac{\underline{\lambda}_{n-2}\cdot\underline{\lambda}_{n-1}}{y^2}\right)\dot{\varphi}_{10}-\underline{\lambda}_{n-2}\cdot\underline{\lambda}_{n-1}\dot{\varphi}_{11}\right)} \\
&\quad \times \sum_{i,j} m_i m_j \langle \mu_j | \left(g^{-1}_{SL(n)_{(k_1,k_2,l)}}\right)^{\#} g^{-1}_{SL(n)_{(k_1,k_2,l)}} |\mu^i\rangle,
\end{aligned}
\tag{9.154}
$$

where b_1 and b_2 are constants that depend on the highest weight representation of E_{n+1}, k_1 and k_2 are the levels with respect to the $\vec{\alpha}_{n+1}$ and $\vec{\alpha}_{n-1}$ simple roots, respectively, and l labels the representations of $SL(n)$ at level (k_{n-1}, k_{n+1}), while $g^{-1}_{A_{(k,l)}(n,n)}$ is the $SL(n)$ coset element acting on a subset of the E_{n+1} lattice states. The splitting of the u into a sum over levels k is a consequence of the matrix element $\langle \mu_j | \left(g^{-1}_{A_{(k,l)}}\right)^{\#} g^{-1}_{A_{(k,l)}} |\mu^i\rangle$ vanishing between states $|\mu^i\rangle$ and $|\mu^j\rangle$ with different H_1 eigenvalues.

One may now construct the automorphic form φ by taking $\Phi_{E_{n+1}} = \sum_{\Lambda} \frac{1}{(\langle\varphi|\varphi\rangle)^s}$ We will evaluate this automorphic form in a similar way to that constructed out of a highest weight E_{n+1} representation with a single deleted node, by splitting the sum over Λ into c sums, where c is the number of $SL(n)$ representations contained in the highest weight representation of E_{n+1} representation, then in all c sums we perform a Poisson resummation over the integers m_i acted on by coset elements $g^{-1}_{SL(n)_{(q_1,q_2)}}$ appearing at all levels lower than the maximum level appearing in that sum. However, since we have deleted two nodes there is an obvious ambiguity in the ordering of the levels (k_1, k_2). We will define an ordering such that $(q_1, q_2) \leq (k_1, k_2)$ if $q_1 + 2q_2 \leq k_1 + 2k_2$, this ordering ensures, after Poisson resummation and a redefinition of the integration variable t (in the same vein as the single deletion procedure) in each sum, that terms with $\hat{m}_i \neq 0$, in the sum over the Poisson resummed lattice, vanish in the large volume limit of the IIA torus $V_{n(A)} \to \infty$ the E_{n+1}. Following this procedure, after some work, one finds

$$
\begin{aligned}
\Phi_{E_{n+1}} = &\sum_{(q_1,q_2)} \sum_{r} \pi^{\frac{1}{2}\sum_{(p_1,p_2)<(q_1,q_2)} M_{(p_1,p_2)}} \frac{\Gamma\left(s-\frac{1}{2}\sum_{(p_1,p_2)<(q_1,q_2)} M_{(p_1,p_2)}\right)}{\Gamma(s)} \\
&\times V_{n(A)}^{2s\left(A-\frac{1}{n}k_1-\frac{2}{n}k_2\right)+\sum_{(p_1,p_2)<(q_1,q_2)} M_{(p_1,p_2)}\left(\frac{1}{n}(q_1-p_1)+\frac{2}{n}(q_2-p_2)\right)} \\
&\quad \times g_{s(A)}^{2s\left(B+\frac{3}{4}k_1-\frac{1}{2}k_2\right)+\sum_{(p_1,p_2)<(q_1,q_2)} M_{(p_1,p_2)}\left(-\frac{3}{4}(q_1-p_1)+\frac{1}{2}(q_2-p_2)\right)} \Phi_{SL(n)_{(q_1,q_2,r)}},
\end{aligned}
\tag{9.155}
$$

where $M_{(p_1,p_2)}$ is the number of states at level (p_1, p_2) in the highest weight representation of E_{n+1} and $\Phi_{SL(n)_{(q_1,q_2,r)}}$ is the automorphic form constructed out of the r'th representation of $SL(n)$ at level (q_1, q_2) in the decomposition of a highest weight representation of E_{n+1}. The constants A and B are related to the highest weight representation used to construct the E_{n+1} automorphic form. Writing the highest weight $\vec{\Lambda}_\Phi$ of the representation used to construct the automorphic form

$\Phi_{E_{n+1}}$ as a linear combination of the fundamental weights of E_{n+1}, $\vec{\Lambda}_{\Phi} = \sum_{k=1}^{n+1} a_k \vec{\Lambda}_k$, where the coefficients a_k are positive integers or zero, gives

$$A = \sum_{k=1}^{n-2} \frac{8k}{n\,(8-n)} a_k + \frac{5n-8}{n\,(8-n)} a_n + \frac{3}{8-n} a_{n-1} + \frac{2}{8-n} a_{n+1} \tag{9.156}$$

and

$$B = \frac{1}{4} a_{n-1} - \frac{1}{4} a_n - \frac{1}{2} a_{n+1}. \tag{9.157}$$

9.6 $\overline{5}$ of $SL(5)$ with Highest Weight $\vec{\Lambda}_{n+1}$

We will examine the automorphic form $\Phi_{E_{n+1}}$ constructed out of the coset element $g \in SL(5)/SO(5)$ in the representation of $SL(5)$ with highest weight $\vec{\Lambda}_{n+1}$. This is the well known automorphic form that appears as the coefficient function of the R^4 and $\partial^4 R^4$ higher derivative terms in $d = 7$ dimensions. Through taking the five limits, discussed in the previous section, we will find conditions under which this automorphic form could exist as a coefficient function for a higher derivative term.

9.6.1 Perturbative Limit

In the perturbative limit we delete node $n+1$, this leaves us with a $GL(1) \times SO(n,n)$ decomposition of the group element $g \in SL(5)/SO(5)$. Table 4 gives the decomposition of this highest weight representation with respect to the $GL(1) \times SO(n,n)$ subgroup and the quantities relevant to the construction of the automorphic form in this limit.

A generic higher derivative term in the $d = 7$ action with coefficient function that is an invariant unconstrained automorphic form constructed out of the representation of $SL(5)$ with highest weight Λ_{n+1} is then

$$\begin{aligned} &\lim_{g_d \to 0} l_s^{l-7} \int d^7x \sqrt{-g_S}\, g_d^{\frac{4\Delta-14}{5}} \Phi_{\Lambda_{n+1}} \mathcal{O}_S \\ &= l_s^{l-7} \int d^7x \sqrt{-g_S} \lim_{g_d \to 0} g_d^{\frac{4\Delta-14}{5}} \left(g_d^{-\frac{8}{5}s} \Phi^s_{\underline{0}(1)} + \pi^{\frac{1}{2}} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} g_d^{\frac{2}{5}s-1} \Phi^{s-\frac{1}{2}}_{\underline{\lambda}_2(4)} \right) \mathcal{O}_S, \end{aligned} \tag{9.158}$$

where $\Phi_{\underline{\lambda}_k(r)}$ is the automorphic form constructed out of the representation of $SO(n,n)$ with highest weight $\underline{\lambda}_k$ and (r) is the dimension of this highest weight representation. Demanding that this generic higher derivative term agrees with a perturbative expansion in g_d gives the conditions

$$-\frac{8}{5}s + \frac{4\Delta-14}{5} = -2 + 2m_0, \tag{9.159}$$

and

$$\frac{2}{5}s - 1 + \frac{4\Delta-14}{5} = -2 + 2m_1, \tag{9.160}$$

(q,r)	(0,1)	(1,1)
SO(n,n)	0	λ_2
d	1	4
$\sum_{p<q} M_p$	0	1
$\sum_{p<q} pM_p$	0	0
$\left(\frac{b}{x^2} - q\right)$	$\frac{4}{5}$	$-\frac{1}{5}$
$\sum_{p<q} (q-p) M_p$	0	1

Table 4: Decomposition of the **5** of $SL(5)$ in the perturbative limit

where $m_0, m_1 \in \mathbb{Z}$. Solving these equations for s and Δ we find

$$s = \frac{1}{2} + m_1 - m_0, \tag{9.161}$$

while

$$\Delta = 2 + 2m_1 + \frac{1}{2}m_0. \tag{9.162}$$

Thus, a higher derivative term in d dimensions may pick up perturbative contributions at m_0 loops and m_1 loops if Δ, the number of inverse space time metrics minus space time metrics in a polynomial of d dimensional curvature R, Cartan forms P and field strengths F, satisfies (9.162). One immediately sees that m_0 must be an even integer, since $\Delta \in \mathbb{Z}$ is a non-negative integer.

These conditions are consistent with the conjectured appearance of this unconstrained invariant automorphic form $\Phi_{\Lambda_{n+1}}$ as the coefficient function of the R^4 term in $d = 7$ type II string theory which picks up tree level and one loop perturbative contributions, in this case one finds $m_0 = 0$, $m_1 = 1$, which gives $s = \frac{3}{2}$ and $\Delta = 4$, as expected. Similarly, the $\partial^4 R^4$ term in $d = 7$ has $\Delta = 6$, from our conditions (9.161), (9.162), the only possible perturbative contributions to a higher derivative term with this automorphic form and $\Delta = 6$ are given at tree level and two loops, this leads to $s = \frac{5}{2}$, which agrees with known results. So, in general, one finds that a higher derivative term in d dimensions that is the product of a polynomial in the curvature R, Cartan forms S and field strengths F contracted with Δ inverse space time metrics minus space time metrics and an invariant unconstrained automorphic form $\Phi_{SL(5)}$ constructed out of the representation of $SL(5)$ with highest weight $\vec{\Lambda}_{n+1}$ is consistent with a perturbative expansion, where it may pick up contributions at m_0 and m_1 loops, for m_0 an even integer, s satisfying (9.161) and Δ satisfying (9.162). It is interesting to note that the $\partial^6 R^4$ term, where $\Delta = 7$, with this automorphic form appears to be consistent with a perturbative expansion if $m_0 = m_1 = 2$ and $s = \frac{1}{2}$.

9.6.2 Large Volume limit of the IIB Torus

In the large volume limit of the IIB torus we delete node n, this leaves us with a $GL(1) \times SL(2) \times SL(n)$ decomposition of the group element $g \in SL(5)/SO(5)$. Table 5 gives the decomposition of this highest weight representation with respect to the $GL(1) \times SL(2) \times SL(n)$ subgroup and the

(q,r)	(0,1)	(1,1)
$SL(2)\times SL(n)$	$(\mu,\underline{0})$	$(0,\lambda_1)$
d	2	3
$\sum_{p<q} M_p$	0	2
$\sum_{p<q} pM_p$	0	0
$\left(\frac{b}{x^2}-q\right)$	$\frac{3}{5}$	$-\frac{2}{5}$
$\sum_{p<q}(q-p)M_p$	0	2

Table 5: Decomposition of the **5** of $SL(5)$ in the IIB volume limit

quantities relevant to the construction of the automorphic form in this limit.

A generic higher derivative term in the $d=7$ action with coefficient function that is an unconstrained Eisenstein-like automorphic form constructed out of the representation of $SL(5)$ with highest weight Λ_{n+1} is then

$$\begin{aligned}&\lim_{V_{n(B)}\to\infty} l_d^{l-7}\int d^7x\sqrt{-g}V_{n(B)}^{-\frac{l-2}{5}}\Phi_{\Lambda_{n+1}}\mathcal{O}\\&=l_d^{l-7}\int d^7x\sqrt{-g}\lim_{V_{n(B)}\to\infty}V_{n(B)}^{\frac{2-l}{5}}\left(V_{n(B)}^{\frac{4}{5}s}\Phi^s_{(\mu,\underline{0})(2)}+\pi^1\frac{\Gamma(s-1)}{\Gamma(s)}V_{n(B)}^{-\frac{8}{15}s+\frac{4}{3}}\Phi^{s-1}_{\left(0,\underline{\lambda}_1\right)(3)}\right)\hat{\mathcal{O}},\end{aligned}\tag{9.163}$$

where a is a non-negative integer, $\Phi_{\left(a\mu,\underline{\lambda}_k\right)(r)}$ is the automorphic form constructed out of the representation of $SL(2)\times SL(n)$ with highest weight $(a\mu,\underline{\lambda}_k)$ and (r) is the dimension of this highest weight representation. The analytic terms in the expansion in powers of $V_{n(B)}$ of this generic higher derivative term, satisfy the following conditions in the large volume limit of the IIB torus $V_{n(B)}\to\infty$,

$$4s-l+2\leq 0,\tag{9.164}$$

and

$$-4s-3l+26<0,\tag{9.165}$$

where l is a non-negative integer. One may substitute for s (from (9.161)) in the above conditions to obtain conditions on the number of derivatives as a function of the loop orders m_0 and m_1 at which this higher derivative term picks up perturbative contributions

$$l\geq 4+4\left(m_1-m_0\right),\tag{9.166}$$

and

$$l>\frac{4}{3}\left(m_0-m_1\right)+8.\tag{9.167}$$

Thus, the large volume limit of the IIB torus $V_{n(B)}\to\infty$ constrains the analytic terms in an expansion of an arbitrary higher derivative term in the $d=7$ effective action of type IIA/B string theory and M-theory in powers of the volume of the type IIB torus $V_{n(B)}$ to satisfy the equations (9.166) and (9.167), relating the loop order m_0 and m_1 of the perturbative contributions to the number of derivatives l in the higher derivative term.

9.6.3 Large Volume Limit of the M-theory Torus

In the large volume limit of the M-theory torus we delete node $n-1$, this leaves us with a $GL(1)\times SL(n+1)$ decomposition of the group element $g\in SL(5)/SO(5)$. Table 6 gives the decomposition of this highest weight representation with respect to the $GL(1)\times SL(n+1)$ subgroup and the quantities relevant to the construction of the automorphic form in this limit.

A generic higher derivative term in the $d=7$ action with coefficient function that is an invariant unconstrained automorphic form constructed out of the representation of $SL(5)$ with highest weight Λ_{n+1} is then

$$\lim_{V_m\to\infty} l_d^{l-7}\int d^7x\sqrt{-g}V_m^{-\frac{l-2}{5}}\Phi_{\Lambda_{n+1}}\mathcal{O} = l_d^{l-7}\int d^{11}x\sqrt{-g} \\ \times \lim_{V_m\to\infty} V_m^{-\frac{l-2}{5}}\left(V_m^{\frac{3}{10}s}\Phi^s_{\underline{\lambda}_3(4)}+\pi^2\frac{\Gamma(s-2)}{\Gamma(s)}V_m^{-\frac{6}{5}s+3}\Phi^{s-2}_{\underline{0}(1)}\right)\hat{\mathcal{O}}, \tag{9.168}$$

where $\Phi_{\underline{\lambda}_k(r)}$ is the automorphic form constructed out of the representation of $SL(n+1)$ with highest weight $\underline{\lambda}_k$ and (r) is the dimension of this highest weight representation. The analytic terms in an expansion in powers of V_m of the unconstrained Eisenstein like higher derivative term constructed from the representation of $SL(5)$ with highest weight $\vec{\Lambda}_{n+1}$ satisfy the following conditions in the large volume limit of the M-theory torus $V_m\to\infty$,

$$\frac{2-l}{5}+\frac{3}{10}s<0, \tag{9.169}$$

and

$$\frac{2-l}{5}-\frac{6}{5}s+3\le 0, \tag{9.170}$$

where l is a non-negative integer. One may substitute for s (from (9.161)) in the above conditions to obtain conditions on the number of derivatives as a function of the loop orders m_0 and m_1 at which this higher derivative term picks up perturbative contributions

$$l>\frac{11}{4}+\frac{3}{2}\left(m_1-m_0\right), \tag{9.171}$$

and

$$l\ge 14-6\left(m_0-m_1\right). \tag{9.172}$$

Thus, the large volume limit of the M-theory torus $V_m\to\infty$ constrains the analytic terms in an expansion of an arbitrary higher derivative term in the $d=7$ effective action of type IIA/B string theory and M-theory in powers of the volume of the M-theory torus V_m to satisfy the equations (9.171) and (9.172), relating the loop order m_0 and m_1 of the perturbative contributions to the

(q,r)	(0,1)	(1,1)
$SL(n+1)$	$\underline{\lambda}_3$	$\underline{0}$
d	4	1
$\sum_{p<q} M_p$	0	4
$\sum_{p<q} pM_p$	0	0
$\left(\frac{b}{x^2} - q\right)$	$\frac{1}{5}$	$-\frac{4}{5}$
$\sum_{p<q} (q-p) M_p$	0	2

Table 6: Decomposition of the **5** of $SL(5)$ in the M-theory limit

number of derivatives l in the higher derivative term.

9.6.4 Decompactification of a Single Dimension Limit

In the decompactification of a single compact dimension limit we delete node 1, this leaves us with a $GL(1) \times SL(2) \times SL(3)$ decomposition of the group element $g \in SL(5)/SO(5)$. Table gives 7 the decomposition of this highest weight representation with respect to the $GL(1) \times SL(2) \times SL(3)$ subgroup and the quantities relevant to the construction of the automorphic form in this limit.

A generic higher derivative term in the $d = 7$ action with a coefficient function that is an invariant unconstrained automorphic form constructed out of the representation of $SL(5)$ with highest weight $\vec{\Lambda}_{n+1}$ is then

$$\lim_{\frac{r_{d+1}}{l_{d+1}}\to\infty} l_{d+1}^{\frac{6}{5}(l-7)} \int d^7x\sqrt{-g}\frac{r_{d+1}}{l_{d+1}}^{-\frac{l-d}{5}} \Phi_{\Lambda_{n+1}}\mathcal{O}$$
$$= \int d^8x\sqrt{-g} \lim_{\frac{r_{d+1}}{l_{d+1}}\to\infty} \frac{r_{d+1}}{l_{d+1}}^{-\frac{l-2}{5}} \left(\frac{r_{d+1}}{l_{d+1}}^{\frac{4}{5}s}\Phi^s_{(0,\underline{\lambda}_2)(3)} + \pi^{\frac{3}{2}}\frac{\Gamma\left(s-\frac{3}{2}\right)}{\Gamma(s)}\frac{r_{d+1}}{l_{d+1}}^{-\frac{6}{5}s+3}\Phi^{s-\frac{3}{2}}_{(\mu,\underline{0})(2)}\right)\hat{\mathcal{O}}, \tag{9.173}$$

where $\Phi_{a\mu,\underline{\lambda}_k(r)}$ is the automorphic form constructed out of the representation of $SL(2)\times SL(3)$ with highest weight $(\mu,\underline{\lambda}_k)$ and (r) is the dimension of this highest weight representation. The analytic terms in an expansion in powers of $\frac{r_{d+1}}{l_{d+1}}$ of the unconstrained Eisenstein-like higher derivative term constructed from the representation of $SL(5)$ with highest weight $\vec{\Lambda}_{n+1}$ satisfy the following conditions in the large volume limit of the M-theory torus $\frac{r_{d+1}}{l_{d+1}} \to \infty$,

$$\frac{2-l}{5} + \frac{4}{5}s \leq 0 \tag{9.174}$$

and

$$\frac{2-l}{5} - \frac{6}{5}s + 3 \leq 0, \tag{9.175}$$

where l is a non-negative integer. One may substitute for s, from (9.161), in the above conditions to obtain conditions on the number of derivatives as a function of the loop orders m_0 and m_1 at

(q,r)	(0,1)	(1,1)
$SL(2)\times SL(3)$	$(0,\underline{\lambda}_2)$	$(\mu,\underline{0})$
d	3	2
$\sum_{p<q} M_p$	0	3
$\sum_{p<q} pM_p$	0	0
$\left(\frac{b}{x^2}-q\right)$	$\frac{2}{5}$	$-\frac{3}{5}$
$\sum_{p<q}(q-p)M_p$	0	3

Table 7: Decomposition of the **5** of $SL(5)$ in the decompactification of a single dimension limit

which this higher derivative term picks up perturbative contributions

$$l \geq 4+4\left(m_1-m_0\right), \tag{9.176}$$

and

$$l \geq 14-6\left(m_1-m_0\right). \tag{9.177}$$

Therefore, the decompactification of a single dimension limit $\frac{r_{d+1}}{l_{d+1}} \to \infty$ constrains the analytic terms in an expansion of an arbitrary higher derivative term in the $d=7$ effective action of type IIA/B string theory and M-theory in powers of the volume of the M-theory torus $\frac{r_{d+1}}{l_{d+1}}$ to satisfy the equations (9.176) and (9.177), relating the loop order m_0 and m_1 of the perturbative contributions to the number of derivatives l in the higher derivative term.

9.6.5 Large Volume Limit of the IIA Torus

In the large volume limit of the IIA torus we delete nodes $n-1$ and $n+1$, this leaves us with a $GL(1)\times GL(1)\times SL(n)$ decomposition of the group element $g\in SL(5)/SO(5)$. Table 8 gives the decomposition of this highest weight representation with respect to the $GL(1)\times GL(1)\times SL(n+1)$ subgroup and the quantities relevant to the construction of the automorphic form in this limit.

A generic higher derivative term in the $d=7$ action with coefficient function that is an invariant unconstrained automorphic form constructed out of the representation of $SL(5)$ with highest weight Λ_{n+1} is then

$$\begin{aligned}
&\lim_{V_{n(A)}\to\infty} l_d^{l-7}\int d^7x\sqrt{-g}V_{n(A)}^{-\frac{l-2}{5}}\Phi_{\Lambda_{n+1}}\mathcal{O} \\
&= l_d^{l-7}\int d^{10}x\sqrt{-g}\lim_{V_{n(A)}\to\infty} V_{n(A)}^{-\frac{l-2}{5}}\left(V_{n(A)}^{\frac{4}{5}s}g_{s(A)}^{-s}\Phi^{s}_{\underline{0}(1)}+\pi^{\frac{1}{2}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(s\right)}V_{n(A)}^{\frac{2}{15}s+\frac{1}{3}}g_{s(A)}^{\frac{1}{2}s-\frac{3}{4}}\Phi^{s-\frac{1}{2}}_{\underline{\lambda}_2(3)}\right. \\
&\qquad\left.+\pi^2\frac{\Gamma\left(s-2\right)}{\Gamma\left(s\right)}V_{n(A)}^{-\frac{6}{5}s+3}g_{s(A)}^{-\frac{1}{2}s+\frac{5}{4}}\Phi^{s-2}_{\underline{0}(1)}\right)\hat{\mathcal{O}},
\end{aligned} \tag{9.178}$$

where $\Phi_{\underline{\lambda}_k(r)}$ is the automorphic form constructed out of the representation of $SL(n)$ with highest weight $\underline{\lambda}_k$ and (r) is the dimension of this highest weight representation. The analytic terms in an expansion in powers of $V_{n(A)}$ of the unconstrained Eisenstein like higher derivative term constructed

(q_1, q_2, r)	(0,0,1)	(1,0,1)	(1,1,1)
$SL(n)$	$\underline{0}$	$\underline{\lambda_2}$	$\underline{0}$
d	1	3	1
$\sum_{(p_1,p_2)<(q_1,q_2)} M_{p_1,p_2}$	0	1	4
$\sum_{(p_1,p_2)<(q_1,q_2)} p_1 M_{(p_1,p_2)}$	0	0	3
$\sum_{(p_1,p_2)<(q_1,q_2)} p_2 M_{(p_1,p_2)}$	0	0	0
$A_{n+1} - \frac{1}{n}q_1 - \frac{2}{n}q_2$	$\frac{2}{5}$	$\frac{1}{15}$	$-\frac{3}{5}$
$B_{n+1} + \frac{3}{4}q_1 - \frac{1}{2}q_2$	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$
$\sum_{(p_1,p_2)<(q_1,q_2)} \frac{1}{n}\left((q_1+2q_2)-(p_1+2p_2)\right) M_{(p_1,p_2)}$	0	$\frac{1}{3}$	3
$\sum_{(p_1,p_2)<(q_1,q_2)} \frac{1}{4}\left((2q_2-3q_1)-(2p_2-3p_1)\right) M_{(p_1,p_2)}$	0	$-\frac{3}{4}$	$\frac{5}{4}$

Table 8: Decomposition of the **5** of $SL(5)$ in the type IIA volume limit

from the representation of $SL(5)$ with highest weight $\vec{\Lambda}_{n+1}$ satisfy the following conditions in the large volume limit of the type IIA torus $V_{n(A)} \to \infty$,

$$\frac{2-l}{5} + \frac{4}{5}s \leq 0, \tag{9.179}$$

$$\frac{2-l}{5} + \frac{2}{15}s + \frac{1}{3} < 0, \tag{9.180}$$

and

$$\frac{2-l}{5} - \frac{6}{5}s + 3 \leq 0, \tag{9.181}$$

where l is a non-negative integer. One may substitute for s in the above conditions to obtain conditions on the number of derivatives as a function of the loop orders m_0 and m_1 at which this higher derivative term picks up perturbative contributions

$$l \geq 4 + 4\left(m_1 - m_0\right), \tag{9.182}$$

$$l > 4 + \frac{2}{3}\left(m_1 - m_0\right), \tag{9.183}$$

and

$$l \geq 12 - 6\left(m_1 - m_0\right), \tag{9.184}$$

So, we see that the large volume limit of the IIA torus $V_m \to \infty$ constrains the analytic terms in an expansion of an arbitrary higher derivative term in the $d = 7$ effective action of type IIA/B string theory and M-theory in powers of the volume of the type IIA torus $V_{n(A)}$ to satisfy the equations (9.182), (9.183) and (9.184), relating the loop order m_0 and m_1 of the perturbative contributions to the number of derivatives l in the higher derivative term.

Transforming to string frame in the $d = 10$ type IIA theory, where the Einstein frame metric g_E is related to the string frame metric g_s by $g_E = e^{\frac{1}{2}\phi}g_s$. One finds that, in the large volume limit of the type IIA torus, a generic higher derivative term in the $d = 7$ type IIA/B or M-theory effective action with an unconstrained Eisenstein-like automorphic form constructed from the **5** of $SL(5)$ coefficient function becomes

$$l_d^{l-7}\int d^{10}x\sqrt{-g}\lim_{V_{n(A)}\to\infty} V_{n(A)}^{-\frac{l-2}{5}} g_{s(A)}^{\frac{\Delta-5}{2}}$$
$$\times\left(V_{n(A)}^{\frac{4}{5}s}g_{s(A)}^{-s}\Phi_{\underline{0}(1)}^{s}+\pi^{\frac{1}{2}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(s\right)}V_{n(A)}^{\frac{2}{15}s+\frac{1}{3}}g_{s(A)}^{\frac{1}{2}s-\frac{3}{4}}\Phi_{\underline{\lambda}_2(3)}^{s-\frac{1}{2}}+\pi^{2}\frac{\Gamma\left(s-2\right)}{\Gamma\left(s\right)}V_{n(A)}^{-\frac{6}{5}s+3}g_{s(A)}^{-\frac{1}{2}s+\frac{5}{4}}\Phi_{\underline{0}(1)}^{s-2}\right)\hat{\mathcal{O}}. \tag{9.185}$$

Equation (9.185) must agree with a perturbative expansion in $g_{s(A)}$. Therefore, for all analytic terms that do not vanish in the large volume limit of the type IIA torus we require

$$\frac{\Delta-5}{2}-s=-2+2k_0 \tag{9.186}$$

and

$$\frac{\Delta-5}{2}-\frac{1}{2}s+\frac{5}{4}=-2+2k_1, \tag{9.187}$$

where k_0 and k_1 are non-negative integers. Note that we expect the second term in equation (9.185) to vanish in the large volume limit of the type IIA torus since it explicitly depends on the torus moduli $\underline{\phi}$.

9.7 133 of E_7 with Highest Weight $\vec{\Lambda}_{n+1}$

We will examine the automorphic form $\Phi_{E_{n+1}}$ constructed out of the coset element $g\in E_7/SU(8)$ in the representation of E_7 with highest weight $\vec{\Lambda}_{n+1}$. This unconstrained automorphic form is not known to be the coefficient function of any higher derivative terms in $d=4$ dimensions. Through taking the perturbative limit, we will see that the unconstrained Eisenstein-like automorphic form constructed from the representation of E_7 with highest weight $\vec{\Lambda}_{n+1}$ does not agree with a perturbative expansion in g_d for any higher derivative term in $d=4$ dimensions.

9.7.1 Perturbative Limit

In the perturbative limit we delete node $n+1$, this leaves us with a $GL(1)\times SO(n,n)$ decomposition of the group element $g\in SL(5)/SO(5)$. Table 9 gives the decomposition of this highest weight representation with respect to the $GL(1)\times SO(n,n)$ subgroup and the quantities relevant to the construction of the automorphic form in this limit.

A generic higher derivative term in the $d=4$ action with coefficient function that is an invariant unconstrained automorphic form constructed out of the representation of E_7 with highest weight

(q,r)	(0,1)	(1,1)	(2,1)	(2,2)	(3,1)	(4,1)
SO(n,n)	0	$\underline{\lambda}_6$	$\underline{0}$	$\underline{\lambda}_2$	$\underline{\lambda}_6$	$\underline{0}$
d	1	32	1	66	32	1
$\sum_{p<q} M_p$	0	1	33	33	100	132
$\sum_{p<q} pM_p$	0	0	32	32	166	262
$\left(\frac{b}{x^2} - q\right)$	2	1	0	0	-1	-2
$\sum_{p<q} (q-p) M_p$	0	1	34	34	134	266

Table 9: Decomposition of the **133** of E_7 in the perturbative limit

$\vec{\Lambda}_{n+1}$ is then

$$\begin{aligned}
&\lim_{g_d\to 0} l_s^{l-4} \int d^4x\sqrt{-g_S}\, g_d^{2\Delta-4}\Phi_{\Lambda_{n+1}}\mathcal{O}_S \\
&= l_s^{l-4} \int d^4x\sqrt{-g_S} \lim_{g_d\to 0} g_d^{2\Delta-4}\Bigg(g_d^{-4s}\Phi^s_{\underline{0}(1)} + \pi^{\frac{1}{2}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} g_d^{-2s-1}\Phi^{s-\frac{1}{2}}_{\underline{\lambda}_6(32)} \\
&\qquad + \pi^{\frac{33}{2}}\frac{\Gamma\left(s-\frac{33}{2}\right)}{\Gamma(s)} g_d^{-34}\left(\Phi^{s-\frac{33}{2}}_{\underline{0}(1)} + \Phi^{s-\frac{33}{2}}_{\underline{\lambda}_2(66)}\right) \\
&\qquad + \pi^{50}\frac{\Gamma(s-50)}{\Gamma(s)} g_d^{2s-134}\Phi^{s-50}_{\underline{\lambda}_6(32)} \\
&\qquad + \pi^{66}\frac{\Gamma(s-66)}{\Gamma(s)} g_d^{4s-266}\Phi^{s-266}_{\underline{0}(1)}\Bigg)\mathcal{O}_S,
\end{aligned} \tag{9.188}$$

where $\Phi_{\underline{\lambda}_k(r)}$ is the automorphic form constructed out of the representation of $SO(n,n)$ with highest weight $\underline{\lambda}_k$ and (r) is the dimension of this highest weight representation. Demanding that this generic higher derivative term agrees with a perturbative expansion in g_d gives the conditions

$$2\Delta - 4 - 4s = -2 + 2m_0, \tag{9.189}$$

$$2\Delta - 4 - 2s - 1 = -2 + 2m_1, \tag{9.190}$$

$$2\Delta - 4 - 34 = -2 + 2m_2, \tag{9.191}$$

$$2\Delta - 4 + 2s - 134 = -2 + 2m_3, \tag{9.192}$$

$$2\Delta - 4 + 4s - 266 = -2 + 2m_4, \tag{9.193}$$

where m_0, m_1, m_2, m_3, m_4 are non-negative integers. Adding equations (9.190) and (9.192) one finds

$$\Delta = \frac{2(m_1+m_3)+139}{4}. \tag{9.194}$$

We then observe that the numerator of the right hand side of equation (9.194) is an odd number for any m_1 and m_3 and therefore is never divisible by 4, since we cannot have a non-integer Δ we see that the unconstrained Eisenstein-like automorphic form constructed from the **133** of E_7 is not an acceptable coefficient function for any higher derivative term in the effective action of type

IIA/B string theory and M-theory in $d = 4$ dimensions.

9.8 248 of E_8 with Highest Weight $\vec{\Lambda}_1$

We will examine the automorphic form $\Phi_{E_{n+1}}$ constructed out of the coset element $g \in E_8/SO(16)$ in the representation of E_8 with highest weight $\vec{\Lambda}_1$. This automorphic form has rarely been considered in the literature and does not appear as the coefficient function of the known R^4, $\partial^4 R^4$ higher derivative terms in $d = 3$ dimensions. We will evaluate the perturbative limit $g_d \to 0$ to find conditions under which this automorphic form could exist as a coefficient function for a higher derivative term.

9.8.1 Perturbative Limit

In the perturbative limit we delete node $n+1$, this leaves us with a $GL(1) \times SO(n,n)$ decomposition of the group element $g \in E_8/SO(16)$. Table 10 gives the decomposition of this highest weight representation with respect to the $GL(1) \times SO(n,n)$ subgroup and the quantities relevant to the construction of the automorphic form in this limit.

A generic higher derivative term in the $d = 3$ action with coefficient function that is an invariant unconstrained automorphic form constructed out of the representation of E_8 with highest weight $\vec{\Lambda}_1$ is then

$$\begin{aligned}
&\lim_{g_d \to 0} l_s^{l-3} \int d^3x \sqrt{-g_S}\, g_d^{4\Delta-6} \Phi_{\Lambda_1} \mathcal{O}_S \\
&= l_s^{l-3} \int d^3x \sqrt{-g_S} \lim_{g_d \to 0} g_d^{4\Delta-6} \Bigg(g_d^{-4s} \Phi^s_{\underline{\lambda}_1(14)} + \pi^7 \frac{\Gamma(s-7)}{\Gamma(s)} g_d^{-(2s+14)} \Phi^{s-7}_{\underline{\lambda}_{n-1}(64)} \\
&\qquad + \pi^{39} \frac{\Gamma(s-39)}{\Gamma(s)} g_d^{-142} \Phi^{s-39}_{\underline{0}(1)} + \pi^{39} \frac{\Gamma(s-39)}{\Gamma(s)} g_d^{-142} \Phi^{s-39}_{\underline{\lambda}_2(91)} \\
&\qquad + \pi^{85} \frac{\Gamma(s-85)}{\Gamma(s)} g_d^{2s-312} \Phi^{s-85}_{\underline{\lambda}_{n-1}(64)} \\
&\qquad + \pi^{117} \frac{\Gamma(s-117)}{\Gamma(s)} g_d^{4s-546} \Phi^{s-117}_{\underline{\lambda}_1(14)} \Bigg) \mathcal{O}_S,
\end{aligned} \tag{9.195}$$

where $\Phi_{\underline{\lambda}_k(r)}$ is the automorphic form constructed out of the representation of $SO(n,n)$ with highest weight $\underline{\lambda}_k$ and (r) is the dimension of this highest weight representation. Demanding that this generic higher derivative term agrees with a perturbative expansion in g_d gives the conditions

$$4\Delta - 6 - 4s = -2 + 2m_0, \tag{9.196}$$

$$4\Delta - 6 - 2s - 14 = -2 + 2m_1, \tag{9.197}$$

$$4\Delta - 6 - 142 = -2 + 2m_2, \tag{9.198}$$

$$4\Delta - 6 + 2s - 312 = -2 + 2m_3, \tag{9.199}$$

(q,r)	(0,1)	(1,1)	(2,1)	(2,2)	(3,1)	(4,1)
SO(n,n)	$\underline{\lambda_1}$	$\underline{\lambda_{n-1}}$	$\underline{0}$	$\underline{\lambda_2}$	$\underline{\lambda_{n-1}}$	$\underline{\lambda_1}$
d	14	64	1	91	64	14
$\sum_{p<q} M_p$	0	14	78	78	170	234
$\sum_{p<q} pM_p$	0	0	14	14	198	390
$\left(\frac{b}{x^2} - q\right)$	2	1	0	0	-1	-2
$\sum_{p<q} (q-p) M_p$	0	14	142	142	312	546

Table 10: Decomposition of the **248** of E_8 in the perturbative limit

$$4\Delta - 6 + 4s - 546 = -2 + 2m_4, \tag{9.200}$$

where $m_0, m_1, ..., m_4$ are non-negative integers. Solving these equations for Δ and s we find

$$\Delta = \frac{73 + m_2}{2} \tag{9.201}$$

while

$$s = \frac{1}{2}\left(71 + m_2 - m_0\right), \tag{9.202}$$

Thus, a higher derivative term in $d = 3$ dimensions may possess the unconstrained Eisenstein-like automorphic form constructed from the 248 of E_8 as a coefficient function if $\Delta \geq 37$ since m_2 is a non-negative integer. The value of s of the automorphic form in this case is given in equation (9.202). This is consistent with the known results for the R^4 and $\partial^4 R^4$ terms in the $d = 3$ effective action, with $\Delta = 4$ and $\Delta = 6$ respectively, that do not possess this automorphic form as a coefficient function. For higher order terms in the $d = 3$ effective action constructed from a number of inverse space time metrics minus space time metrics Δ satisfying $\Delta \geq 37$ one could find that this automorphic form appears as the coefficient function. It should be noted that very little is known about higher derivative terms with $\Delta \geq 37$ therefore it is not clear how the remaining decompactification limits considered in this chapter could lead to further conditions on higher derivative terms that could possess this automorphic form as a coefficient function.

9.9 Conclusion

In this chapter we have shown that the parameters in the effective actions of type IIA/B string and M-theory dimensionally reduced on a torus to $d = 10 - n$ dimensions are naturally associated with certain nodes of the Dynkin diagram that encodes the E_{n+1} symmetry. In particular, the d dimensional coupling g_d depends purely on the E_{n+1} Chevalley field $\dot{\varphi}_{10}$, see equation (9.42), corresponding to node $n+1$ in the E_{n+1} Dynkin diagram given in figure 23. While from equations (9.41) and (9.54) we see that the volume of the torus from the type IIB and M-theory perspectives corresponds to node n and $n-1$ respectively. Equation (9.66) demonstrates that the volume of the torus from the type IIA view point may be written as the exponential of a linear combination of the E_{n+1} Chevalley fields $\dot{\varphi}_{10}$ and $\dot{\varphi}_{11}$, corresponding to nodes $n-1$ and $n+1$. The ratio of the radius of the torus in the $d+1$ direction r_{d+1} to the d dimensional Planck length may be written in terms of the E_{n+1} Chevalley field $\dot{\varphi}_{d+1}$, as we from equation (9.43), and corresponds to node 1.

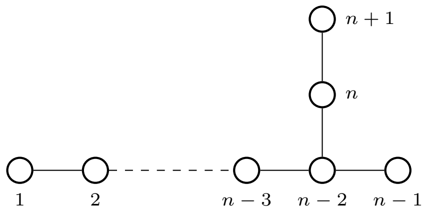

Figure 23: Dynkin diagram for E_{n+1}

The parameters in the effective actions of type IIA/B string and M-theory in $d = 10 - n$ dimensions can therefore be given a group theoretical interpretation.

It follows that for a limit in a given parameter one naturally deletes the node (or nodes) corresponding to this parameter to decompose the E_{n+1} algebra in terms of the resulting subalgebra. The E_{n+1} automorphic forms are then expected to reduce to an expansion in this parameter with coefficient functions that are automorphic forms transforming under the group generated by the subalgebra in this limit. Higher derivative terms in the effective action of type IIA/B string theory are then required to agree with the known properties of type IIA/B string theory and M-theory in each of these limits.

The interpetation of the Chevalley fields of E_{n+1} in terms of the parameters in the effective actions of type IIA/B string and M-theory in $d = 10 - n$ dimensions allows us to write down a general expression for the unconstrained Eisenstein-like automorphic form in each limit. The analysis of the unconstrained Eisenstein-like automorphic form constructed from the representation of $SL(5)$ with highest weight $\vec{\Lambda}_{n+1}$ demonstrates that it is possible that this automorphic form could appear as a coefficient function for higher derivative terms satisfying certain conditions in the $d = 7$ effective action beyond the known R^4 and $\partial^4 R^4$ terms. The weak coupling limit $g_d \to 0$ of the unconstrained Eisenstein-like automorphic form constructed from the representation of E_7 with highest weight $\vec{\Lambda}_{n+1}$ shows that this automorphic form is not compatible with a perturbative expansion in g_d for any higher derivative term and therefore can not appear as the coefficient function for any higher derivative term in $d = 4$ dimensions. The weak coupling limit $g_d \to 0$ of the unconstrained Eisenstein-like automorphic form constructed from the representation of E_8 with highest weight $\vec{\Lambda}_1$ suggests that this automorphic form could be a valid coefficient function for terms at a high enough order in the effective action, in particular for those terms satisfying equation (9.201).

A Formulae and Identities

This section of the appendix contains additional formulae for performing the analysis of the automorphic forms constructed in chapters 8 and 9 and a derivation of the identities used in chapter 5.

A.1 Formulae

Poisson resummation formula:

$$\sum_{\vec{m}\in\mathbb{Z}^N} e^{-\pi(\vec{m}-\vec{a})\cdot A(\vec{m}-\vec{a})+2\pi i\vec{m}\cdot\vec{b}} = \sum_{\hat{\vec{m}}\in\mathbb{Z}^N} \det\mathrm{A}^{-\frac{1}{2}} e^{-\pi(\hat{\vec{m}}+\vec{b})\cdot A^{-1}(\hat{\vec{m}}+\vec{b})+2\pi i(\hat{\vec{m}}+\vec{b})\cdot\vec{a}}, \tag{A.1}$$

where $\vec{m}$, $\vec{a}$, $\vec{b}$, $\hat{\vec{m}}$ are N vectors with integer entries and A is an $N\times N$ matrix.

An integral representation of the function u^{-s} is given by

$$\frac{1}{u^s} = \frac{\pi^s}{\Gamma(s)}\int_0^\infty \frac{dt}{t^{1+s}} e^{-\frac{\pi u}{t}}. \tag{A.2}$$

The Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{A.3}$$

for $s\in\mathbb{C}$. An integral identity of the Bessel function K_λ is

$$\int_0^\infty \frac{dt}{t^{1+\lambda}} e^{-at-b/t} = 2\left|\frac{a}{b}\right|^{\lambda/2} K_\lambda(2\sqrt{|ab|}). \tag{A.4}$$

The asymptotic behaviour of a Bessel function K_λ as $z\to\infty$ is

$$K_\lambda(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{l=0}^{\infty} \frac{1}{(2z)^l} \frac{\Gamma(\lambda+l+\frac{1}{2})}{\Gamma(l+1)\Gamma(\lambda-l+\frac{1}{2})}. \tag{A.5}$$

A.2 Identities

Let B_{ij} and C_{ij} be the components of the dimensionally reduced NS-NS and R-R type IIB two-form gauge fields, transforming under $SL(3,\mathbb{R})$ as

$$B_{ij} \to U(g_0)B_{ij} = D\left(g_0\right)_i{}^k D\left(g_0\right)_j{}^l B_{kl}, \tag{A.6}$$

$$C_{ij} \to U(g_0)C_{ij} = D\left(g_0\right)_i{}^k D\left(g_0\right)_j{}^l C_{kl}, \tag{A.7}$$

and A^i be the components of the graviphoton field which transform under $SL(3,\mathbb{R})$ as

$$A^i \to U(g_0)A^i = D\left(g_0^{-1}\right)^i{}_j A^j, \tag{A.8}$$

where the group elements $D(g_0)$ are in the **3** of $SL(3,\mathbb{R})$ and Roman letters denote three dimensional $SL(3,\mathbb{R})$ indices. Defining $B^m = \frac{1}{2}\epsilon^{mrs}B_{rs}$, $C^n = \frac{1}{2}\epsilon^{mrs}C_{rs}$ and $A_{mn} = \epsilon_{mni}A^i$, where ϵ^{mrs} are the components of the completely antisymmetric tensor ϵ and ϵ_{mni} are the components of ϵ lowered with the $SL(3,\mathbb{R})$ metric on the torus g.

For the first identity observe that

$$\begin{aligned} A^m B_{lm} =& \frac{1}{2}\epsilon^{mrs}A_{rs}B_{lm} \\ =& \frac{1}{2}\epsilon^{mrs}\epsilon_{lmn}A_{rs}B^n \\ =& -A_{ln}B^n, \end{aligned} \tag{A.9}$$

where we have used the general identity $\epsilon^{i_1\ldots i_q k_1\ldots k_p}\epsilon_{j_1\ldots j_q k_1\ldots k_p} = p!q!\delta^{i_1\ldots i_q}_{j_1\ldots j_q}$ for the product of two completely antisymmetric $n = p+q$ tensors with p contracted indices and $\delta^{i_1\ldots i_1}_{j_1\ldots j_q} = \delta^{[i_1}_{[j_1}\ldots\delta^{i_q]}_{j_q]}$.

For the second, consider the object

$$\epsilon_{lmn}A^l B^m C^n. \tag{A.10}$$

Expanding B^m, as defined above, we obtain

$$\begin{aligned} \epsilon_{lmn}A^l B^m C^n =& \frac{1}{2}\epsilon_{lmn}\epsilon^{mrs}A^l B_{rs}C^n \\ =& -\frac{1}{2}\delta^{rs}_{ln}A^l B_{rs}C^n \\ =& -\frac{1}{2}A^l B_{ln}C^n. \end{aligned} \tag{A.11}$$

Expanding C^n, one finds

$$\begin{aligned} \epsilon_{lmn}A^l B^m C^n =& \frac{1}{2}\epsilon_{lmn}\epsilon^{nrs}A^l B^m C_{rs} \\ =& \frac{1}{2}\delta^{rs}_{lm}A^l B^m C_{rs} \\ =& \frac{1}{2}A^l B^m C_{lm}. \end{aligned} \tag{A.12}$$

Therefore, we have

$$\begin{aligned} A^l B^m C_{lm} =& 2\epsilon_{lmn}A^l B^m C^n \\ =& -A^l B_{ln}C^n. \end{aligned} \tag{A.13}$$

Note that these identities also hold for similarly defined gauge fields transforming under $SO(3,\mathbb{R})$.

B Lie Groups, Lie Algebras and Non-linear Representations

Lie groups and their Lie algebras play a prominent role in describing the symmetries and dualities of string theory. This section of the appendix outlines the basic theory and central results of Lie groups, Lie algebras and their representations, relevant to our methods of investigating the effective actions of type IIA/B string theory and M-theory. In particular, we highlight the classical and exceptional Lie algebras that generate the U-duality group of dimensionally reduced type IIA/B string theory and M-theory and explain how one may formulate these theories as non-linear realisations of the U-duality group. The reader may consult references [88, 89] for a more complete treatment of the topic.

B.1 Group Theory

A **group** G is a set of elements and a binary composition law $\circ : G \times G \to G$, known as group multiplication, that satisfies the following axioms:

- Closure - Group multiplication is closed, that is, for all $a, b \in G$, $a \circ b \in G$.
- Associativity - Group multiplication is associative, for all $a, b, c \in G$,

$$(a \circ b) \circ c = a \circ (b \circ c) = a \circ b \circ c. \tag{B.1}$$

- Identity - There exists an identity element $e \in G$, such that, for all $a \in g$,

$$a \circ e = e \circ a = a. \tag{B.2}$$

- Inverse - For every element $a \in G$, there is an an inverse element, $a^{-1} \in G$ such that,

$$a \circ a^{-1} = a^{-1} \circ a = e. \tag{B.3}$$

Intuitively one may think of a Lie group as a continuous group that is also an n dimensional differentiable manifold. The following definition formalises this notion. A **Lie group** G is a differentiable manifold with a group structure such that the group multiplication operation $\circ$: $G \times G \to G$, $(a, b) \to (a \circ b)$, and the group inverse operation $^{-1} : G \to G$, $g \to g^{-1}$, are differentiable. One should note that group multiplication need not be commutative. In general, for $a, b \in G$, one has $a \circ b \neq b \circ a$. A group that satisfies $a \circ b = b \circ a$ for all $a, b \in G$ is known as an **abelian** group. A group for which $a \circ b \neq b \circ a$ for any elements $a, b \in G$ is a **non-abelian** group. A trivial example is the set $\mathbb{R}$ with group multiplication taken to be addition. Since the group multiplication and inverse operations are differentiable and the addition of any two real numbers is a commutative operation, $\mathbb{R}$ is an abelian Lie group when we define the group multiplication operation to be addition.

B.1.1 Subgroups and Cosets

One may also consider the restriction of a group to a subset of the elements in G with the same multiplication law, this leads us to the concept of a subgroup. A **subgroup** H of a group G is a subset of G that satisfies the group axioms under the same group multiplication law as G. The identity element e for a subgroup H of G is necessarily the same identity element as that for the group G. For any group G, the subgroup e and G always exist and are **trivial subgroups**. If a subgroup H of G is neither e nor G then it is a **proper subgroup**. A subgroup H of a group G is a **normal subgroup** of G, if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.

Given a subgroup H of G one may define a **coset**. Let G be a group and H a subgroup of G. For $a, b \in G$, define an equivalence relation $\sim$ by, $a \sim b$ if there exists an $h \in H$ such that $a \circ h = b$ the equivalence class $[a] = \{a \circ h | h \in H\}$ is a **(right) coset**. One may similarly define a **(left) coset** by taking the subgroup element $h \in H$ to act on $a \in G$ from the left, the equivalence class $[a] = \{h \circ a | h \in H\}$ is then a **(left) coset**. If a group G is abelian then left and right cosets of G are equivalent. The set of all cosets for a given group G and subgroup H is a **quotient space**, denoted G/H. In general, for a group G and an arbitrary subgroup H the quotient space G/H does not possess a group structure. The quotient space G/H constructed from a group G and a normal subgroup H is a **quotient group** with a group multiplication operation given by $[a] \circ [b] = [ab]$, for $[a], [b] \in G/H$.

B.1.2 Representations of a Group

So far we have considered groups at an abstract level, however in most situations one is interested in a representation of a group acting on a set. A **representation** of a group G is a homomorphism $\phi : G \to S$ from G to a set of maps S on a set X. By definition, the homomorphism property of the representation preserves the group structure of G that is, if $a, b \in G$ and $\phi : G \to S$ is a representation of G, then $\phi(a) \circ \phi(b) = \phi(a \circ b)$. A representation $\phi : G \to S$ of a group G is a **faithful** representation if ϕ is an isomorphism. While a representation $\phi : G \to S$ of a group G is an **unfaithful** representation if ϕ is a many-to-one homomorphism. A **linear representation** of a group G is a homomorphism $\phi : G \to S$ where the set S of maps on X are linear maps. Most groups appear as linear representations in physics. For example, the gauge fields of type IIA/B string theory and M-theory dimensionally reduced to $d = 10 - n$ dimensions transform as linear representations of the $E_{n+1}(\mathbb{Z})$ U-duality group.

A particular case of interest to us is when a group G is a matrix group. A **matrix group** is a group G where elements of the group are invertible matrices M over a field $\mathbb{K}$ and group multiplication is given by matrix multiplication. A simple example of a matrix group is the general linear group $GL(n, \mathbb{R})$, defined to be the set of $n \times n$ invertible matrices with real entries. In fact all matrix groups are a subgroup of some general linear group. Others we will encounter in this thesis are the special linear group $SL(n, \mathbb{R})$ of $n \times n$ dimensional matrices with real entries, satisfying $det(g) = 1$, for all $g \in SL(n, \mathbb{R})$ and the orthogonal group $O(n, \mathbb{R})$ of $n \times n$ dimensional matrices with real entries, with all elements $g \in O(n, \mathbb{R})$ satisfying $g \circ g^T = \mathbb{I}$, where g^T is the transpose of g.

Note that in the rest of this thesis, the group multiplication operation $\circ : G \times G \to G$ is denoted by $g_1 \circ g_2 = g_1 g_2$ for any $g_1, g_2 \in G$.

B.2 Lie Groups and Lie Algebras

Our principal interests are the classical and exceptional Lie groups. These are matrix Lie groups which are closed subroups of $GL(n, \mathbb{C})$ for some n. In particular we will investigate these Lie groups at the level of the Lie algebra, as one often does. A **Lie algebra** $\mathfrak{g}$ over a field $\mathbb{K}$ is a vector space with a Lie bracket operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the following properties

1. Antisymmetry - The Lie bracket is antisymmetric for all $X, Y \in \mathfrak{g}$

$$[X, Y] = -[Y, X]. \tag{B.4}$$

2. Bilinearity - The Lie bracket is bilinear for all $X, Y, Z \in \mathfrak{g}$ and $a, b \in \mathbb{K}$

$$\begin{aligned} [X, aY + bZ] &= a[X, Y] + b[X, Z], \\ [aY + bZ, X] &= a[Y, X] + b[Z, X]. \end{aligned} \tag{B.5}$$

3. The Jacobi identity - The Jacobi identity holds for all $X, Y, Z \in \mathfrak{g}$

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \tag{B.6}$$

For an **abelian** Lie algebra, $[X_i, X_j] = 0$ for all $X_i, X_j \in \mathfrak{g}$. The **rank** of a Lie algebra is the number of linearly independent elements in the maximal set of commuting generators $\mathfrak{h}$, satisfying $[X_i, X_j] = 0$, for all $X_i \in \mathfrak{h}$, in other words the rank of a Lie algebra $\mathfrak{g}$ is the number of elements in the maximal abelian subalgebra. In this thesis we will restrict out attention to the case where the field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. To relate Lie algebras to Lie groups we will need to define the exponential of a matrix. If A is an $n \times n$ matrix with real or complex entries then the exponential of A is defined by

$$e^A = \sum_{N=0}^{\infty} \frac{1}{N!} A^N. \tag{B.7}$$

one may show that this series always converges. The Lie algebra $\mathfrak{g}$ of a matrix Lie group G is then the set of all matrices X satisfying $e^{tX} \in G$, for all $t \in \mathbb{R}$. So, we find that exponentiating a Lie algebra $X \in \mathfrak{g}$ element of a matrix Lie group G gives a group element in G, this is known as the exponential map. However, one should note that the exponential map from a Lie algebra to a matrix Lie group is not necessarily one-to-one or onto.

At an abstract level the Lie algebra $\mathfrak{g}$ of a Lie group G is simply a set of elements $X \in \mathfrak{g}$, with a Lie bracket operation that satisfies the antisymmetry and bilinearity axioms along with the Jacobi identity. In the same vein as taking a representation of a group, one may similarly consider representations of a Lie algebra. A **representation** of a Lie algebra $\mathfrak{g}$ is a homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}_n$ from $\mathfrak{g}$ to a set of maps $\mathfrak{gl}_n$ on an n dimensional vector space V. The general linear algebra $\mathfrak{gl}_n$ is the set of all linear operators on an n dimensional vector space V.

B.2.1 The Killing Metric

Any Lie algebra $\mathfrak{g}$ with elements $X_i \in \mathfrak{g}$, $i = 1, .., d$ where d is the dimension of the Lie algebra, satisfies

$$[X_i, X_j] = \sum_k f_{ij}{}^k X_k, \tag{B.8}$$

for some **structure constants** $f_{ij}{}^k$. The structure constants are antisymmetric in their subscript indices $f_{ij}^k = -f_{ji}{}^k$ and are constrained by the Jacobi identity which leads to

$$f_{jk}{}^l f_{il}{}^m + f_{ki}{}^l f_{jl}{}^m + f_{ij}{}^l f_{kl}{}^m = 0, \tag{B.9}$$

these properties are both inherited from the definition of the structure constants in terms of the Lie algebra commutator. If the structure constants for a complex Lie algebra with a given basis are real then one has a real form of a complex Lie algebra. In general there may be more than one real form of a complex Lie algebra. For instance, the Lie algebras $\mathfrak{sl}(n, \mathbb{R}) = \{X \in \mathfrak{gl}\,(n, \mathbb{R}) : TrX = 0\}$ and $\mathfrak{su}(n, \mathbb{R}) = \{X \in \mathfrak{sl}\,(n, \mathbb{C}) : X^* + X = 0\}$ are both real forms of the complex Lie algebra $\mathfrak{sl}(n, \mathbb{C}) = \{X \in \mathfrak{gl}\,(n, \mathbb{C}) : TrX = 0\}$ for $n \geq 2$, where X^* is the Hermitian conjugate of X.

Having defined the structure constants of a Lie algebra $\mathfrak{g}$ one may look to construct a metric on this algebra. First, for $X \in \mathfrak{g}$ consider the adjoint map $X \to adX$, which for all $Y \in \mathfrak{g}$, is defined by

$$X \to adX(Y) := [X, Y]\,. \tag{B.10}$$

The adjoint map, takes any element in the Lie algebra $\mathfrak{g}$ and returns a linear transformation on $\mathfrak{g}$ defined through the Lie bracket. In fact the set of maps of the form adX, for $X \in \mathfrak{g}$ form a representation of the Lie algebra $\mathfrak{g}$ known as the adjoint representation. To confirm that this is indeed a representation of $\mathfrak{g}$ one must check that the set of maps adX, for all $X \in \mathfrak{g}$, preserve the commutation relations of the Lie algebra $\mathfrak{g}$. Suppose $X, Y, Z \in \mathfrak{g}$ satisfy $[X, Y] = Z$, for any $U \in \mathfrak{g}$

we then have

$$\begin{aligned}[adX, adY](U) &= [X,[Y,U]] - [Y,[X,U]] \\ &= [X,[Y,U]] + [Y,[U,X]] \\ &= -[U,[X,Y]] \\ &= [[X,Y],U] \\ &= [Z,U] \\ &= adZ(U),\end{aligned} \tag{B.11}$$

where we have made use of the Jacobi identity $(B.6)$. So, the set of maps adX, for all $X \in \mathfrak{g}$, preserve the commutation relations of the Lie algebra $\mathfrak{g}$ and thus form a representation of $\mathfrak{g}$.

The adjoint representation of a Lie algebra $\mathfrak{g}$ may be used to define a metric on $\mathfrak{g}$. For a finite dimensional Lie algebra $\mathfrak{g}$, the map $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, given by

$$B(X_i, X_j) = \frac{1}{a} Tr\,(adX_i, adX_j)\,, \tag{B.12}$$

for $a \in \mathbb{C}$, is the **Killing metric** on $\mathfrak{g}$. The constant a may be chosen to simplify expressions which corresponds to a scaling of the elements of the Lie algebra $\mathfrak{g}$. The Killing metric is independent of the basis chosen for the adjoint representation of $\mathfrak{g}$. For any Lie algebra $\mathfrak{g}$, one may write the components B_{ij} of the Killing metric on $\mathfrak{g}$ in terms of the structure constants ${f_{ij}}^k$. The components of the Killing metric then take the form,

$$B_{ij} = {f_{ik}}^l {f_{jl}}^k. \tag{B.13}$$

B.2.2 Simple and Semi-Simple Lie Algebras

To define the simple and semi-simple Lie algebras, that contain the Lie algebras of the classical and exceptional Lie groups, we must first tie down the concept of an ideal of a Lie algebra. An **ideal** $\mathfrak{h}$ in $\mathfrak{g}$ is a subspace satisfying $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. Consider a finite dimensional Lie algebra $\mathfrak{g}$, defining $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and for all $n \in \mathbb{Z}$, satisfying $j > 1$,

$$\mathfrak{g}^{n+1} = [\mathfrak{g}^n, \mathfrak{g}^n]\,, \tag{B.14}$$

we find a decreasing sequence

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq ... \tag{B.15}$$

known as the **commutator series** of $\mathfrak{g}$. One may show that each $\mathfrak{g}^n$ is an ideal in $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is **solvable** if $\mathfrak{g}^n = 0$ for some n.

A finite dimensional Lie algebra is **simple** if it is not an Abelian algebra and $\mathfrak{g}$ has no proper nonzero ideals. A finite dimensional Lie algebra $\mathfrak{g}$ is **semi-simple** if $\mathfrak{g}$ has no nonzero solvable ideals.

One may show that for any semi-simple Lie algebra $\mathfrak{g}$ the Killing metric B satisfies $det(B_{ij}) \neq 0$, it follows that the inverse Killing metric B^{-1} with components B^{ij} exists and therefore B_{ij} and B^{ij} may be used to raise and lower Lie algebra indices.

B.2.3 Roots of a Lie Algebra

For a semi-simple Lie algebra $\mathfrak{g}$ one may split the generators X_i into two distinct sets. To begin with, for a rank n Lie algebra, one may find a maximal set of n commuting generators, known as the **Cartan sub-algebra** $\mathfrak{h}$ of $\mathfrak{g}$. The generators in the Cartan sub-algebra $H_i \in \mathfrak{h}$, $i = 1, ..., n$, satisfy

$$[H_i, H_j] = 0, \tag{B.16}$$

for all $H_i, H_j \in \mathfrak{h}$. The remaining generators, which we shall denote $E_{\vec{\alpha}}$, have the following commutation relations with an element of the Cartan subalgebra H_i,

$$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}, \tag{B.17}$$

where $\alpha_i \in \mathbb{C}$. Thus, for any generator E_α one finds a corresponding complex number α_i for each element of the Cartan subalgebra H_i, given by the commutation relation $(B.17)$. Therefore, the generators $E_{\vec{\alpha}}$ are eigenstates under the Lie bracket operation, for any element of the Cartan subalgebra H_i, $i = 1, ..., n$, with eigenvalue α_i. One may think of $\vec{\alpha}$ as a vector in an n dimensional space, with components given by α_i, $i = 1, ..., n$ in the following way

$$\vec{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) \, . \tag{B.18}$$

The vector $\vec{\alpha}$ is known as a **root** of the Lie algebra $\mathfrak{g}$. For each element of the Lie algebra $X_i \in \mathfrak{g}$, outside of the Cartan subalgebra, there exists a unique root $\vec{\alpha}$, furthermore, no roots of the Lie algebra are identically zero. Denoting the set of roots $\vec{\alpha}$ of a Lie algebra $\mathfrak{g}$ by Δ, one may split Δ into two sets by defining an arbitrary ordering on the root space. By way of example, one may take a root $\vec{\alpha}$ to be positive, written $\vec{\alpha} > 0$ if its first non-zero component α_i is positive. It is important to stress that there exist alternative orderings of the root space, for instance the root space of a rank n Lie algebra with corresponding simple roots $\vec{\alpha}_i$, $i = 1, 2, ...n$, could be ordered by defining $\vec{\alpha}_1 > \vec{\alpha}_2 > ... > \vec{\alpha}_n$. All roots $\alpha \in \Delta$ that are not positive are then defined to be negative $\vec{\alpha} < 0$. The set of positive roots are then $\Delta_+ = \{\vec{\alpha} \in \Delta | \vec{\alpha} > 0\}$ while the set of negative roots are $\Delta_- < 0$. We then define the set of positive and negative root generators as $\mathfrak{g}_+ = \{E_{\vec{\alpha}} \in \mathfrak{g} : \vec{\alpha} \in \Delta_+\}$ and $\mathfrak{g}_- = \{E_{\vec{\alpha}} \in \mathfrak{g} : \vec{\alpha} \in \Delta_-\}$, respectively. A **simple root** of a semi-simple Lie algebra $\mathfrak{g}$ is a root that cannot be written as the sum of two positive roots. A rank n Lie algebra contains n simple roots that may be used as a basis for the root vector space.

The full set of commutation relations between elements in $\mathfrak{g}$ are

$$[H_i, H_j] = 0, \tag{B.19}$$

$$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}, \tag{B.20}$$

$$\left[E_{\vec{\alpha}}, E_{\vec{\beta}}\right] = \mathcal{N}_{\vec{\alpha},\vec{\beta}} E_{\vec{\alpha}+\vec{\beta}}, \tag{B.21}$$

$$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \sum_i \vec{\alpha}_i H_i, \tag{B.22}$$

where $\mathcal{N}_{\vec{\alpha},\vec{\beta}}$ is a constant that is dependent on the normalisation of the generators. The first two commutation relations have been discussed previously. From the third commutation relation $(B.21)$ we see that the generator arising from commuting elements $E_{\vec{\alpha}} \in \mathfrak{g}$ and $E_{\vec{\beta}} \in \mathfrak{g}$ has eigenvalue $\alpha_i + \beta_i$ under the Lie bracket operation with a Cartan sub-algebra element H_i. The fourth commutation relation $(B.22)$ implies that taking the Lie bracket of two generators with roots $\vec{\alpha}$ and $-\vec{\alpha}$, of opposite sign, returns a linear sum of Cartan subalgebra generators weighted by the root $\vec{\alpha}$.

One may view any root $\vec{\alpha}$ as a linear functional on the Cartan subalgebra, $\alpha : H \to \mathbb{C}$, where the action of α on the basis elements H_i, $i = 1, 2, ..., n$ of the Cartan subalgebra is defined by $\alpha(H_i) = \alpha_i$. Therefore the set of roots $\vec{\alpha} \in \Delta$ are in the dual vector space H^*. The Killing metric may be used to associate a unique element of the Cartan subalgebra H to every element of the dual vector space H^* by defining

$$\alpha(h) = B(h_{\vec{\alpha}}, h), \tag{B.23}$$

where $h \in H$ and $h_{\vec{\alpha}}$ is the element of the Cartan subalgebra dual to $\vec{\alpha}$. To find the element of the Cartan subalgebra $h_{\vec{\alpha}}$ associated with the root $\vec{\alpha}$ we will evaluate $(B.23)$ by taking $h_{\vec{\alpha}} = d^i H_i$ and $h = c^i H_i$ one then has

$$\vec{\alpha}(h) = c^i \alpha_i, \tag{B.24}$$

equation $(B.23)$ then gives

$$\begin{aligned} c^i \alpha_i &= d^i c^j B\left(H_i, H_j\right) \\ &= d^i c^j G_{ij}, \end{aligned} \tag{B.25}$$

where G_{ij} are the components of the the Killing metric restricted to the Cartan subalgebra. Therefore we find $d^i = G^{ij}\alpha_j$ and so the unique element of the Cartan subalgebra H associated with the root $\vec{\alpha}$ is $h_{\vec{\alpha}} = G^{ij}\alpha_j H_i$.

It can then be shown that there exists a scalar product on the space of roots $(\cdot, \cdot) : \Delta \times \Delta \to \mathbb{C}$, which, for $\vec{\alpha}, \vec{\beta} \in \Delta$, is defined by

$$\left(\vec{\alpha}, \vec{\beta}\right) := B\left(h_{\vec{\alpha}}, h_{\vec{\beta}}\right). \tag{B.26}$$

In terms of the components of the roots $\vec{\alpha}$, $\vec{\beta}$ and the Killing metric restricted to the Cartan subalgebra the scalar product on the space of roots is given by

$$\begin{aligned}\left(\vec{\alpha}, \vec{\beta}\right) &= G^{ij}\alpha_i\beta_j \\ &= \alpha_i\beta^i.\end{aligned} \tag{B.27}$$

A Lie algebra is **simply laced** if all of its roots $\vec{\alpha} \in \Delta$ are the same length, as defined by the scalar product $(B.27)$. We will normalise all roots $\vec{\alpha} \in \Delta$ of a simply laced algebra, such that $(\vec{\alpha}, \vec{\alpha}) = 2$.

B.2.4 The Cartan Matrix and Dynkin Diagrams

Given a rank n semi-simple Lie algebra $\mathfrak{g}$ and its simple roots $\vec{\alpha}_i$, $i = 1, ..., n$, one may form the **Cartan matrix** A with components A_{ij} given by

$$A_{ij} = 2\frac{(\vec{\alpha}_i, \vec{\alpha}_j)}{(\vec{\alpha}_i, \vec{\alpha}_i)}. \tag{B.28}$$

The Cartan matrix A has the following properties

- The components of the Cartan matrix A are restricted to the integers $\{-3, -2, -1, 0, 2\}$.
- For the diagonal components of the Cartan matrix A one has $A_{ii} = 2$.
- For $i \neq j$, $A_{ij} \leq 0$.
- A component of the Cartan matrix A_{ij} satisfies $A_{ij} = 0$ if and only if $A_{ji} = 0$.
- There exists a diagonal matrix D with positive entries such that DAD^{-1} gives a symmetric, positive definite quadratic form.

A Cartan matrix A of a rank n Lie algebra $\mathfrak{g}$ is neatly summarised through the use of a **Dynkin diagram**. To construct a Dynkin diagram from a Cartan matrix A of a rank n Lie algebra one draws n nodes, then connects nodes i and j by $A_{ij}A_{ji}$ edges, if $A_{ij} > A_{ji}$ then an $>$ symbol is included along the edges connecting nodes i and j. The Dynkin diagrams may be used to completely classify the finite dimensional Semi-simple Lie algebras. The result is the classical Lie algebras $A_n = \mathfrak{sl}(n+1, \mathbb{C})$, for $n = 1, 2, 3, ...$; $B_n = \mathfrak{o}(2n+1, \mathbb{C})$, for $n = 2, 3, 4, ...$; $C_n = \mathfrak{sp}(n, \mathbb{C})$, for $n = 3, 4, 5, ...$; $D_n = \mathfrak{o}(2n, \mathbb{C})$, for $n = 4, 5, 6, ...$, and the exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8, with Dynkin diagrams given in figures 24 and 25.

Note that a dashed line indicates a line of nodes with each node connected to the two adjacent nodes by a single edge.

To construct the classical and exceptional Lie algebras from their Dynkin Diagram or, equivalently, Cartan matrix one may define the Chevalley-Serre generators by scaling the generators of

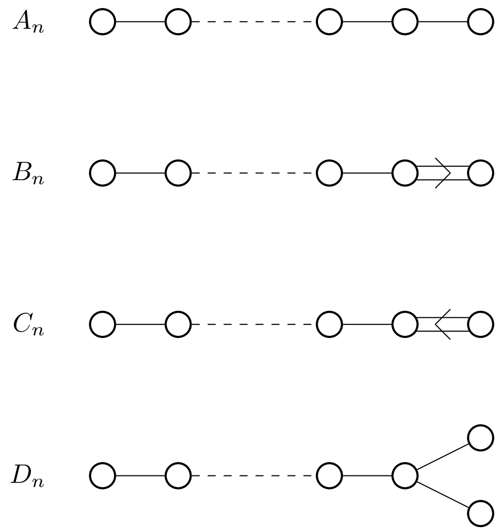

Figure 24: The classical Lie algebras

the Lie algebra $\mathfrak{g}$ in Cartan-Weyl basis satisfying the commutation relations in the following way,

$$
\begin{aligned}
E_a &= \sqrt{\frac{2}{(\vec{\alpha}_a, \vec{\alpha}_a)}} E_{\vec{\alpha}_a}, \\
F_a &= \sqrt{\frac{2}{(\vec{\alpha}_a, \vec{\alpha}_a)}} E_{-\vec{\alpha}_a}, \\
H_a &= 2\frac{\alpha_a^i H_i}{(\vec{\alpha}_a, \vec{\alpha}_a)},
\end{aligned}
\tag{B.29}
$$

where α_a^i is component i of simple root a, H_i are the Cartan subalgebra generators, $E_{\vec{\alpha}_a}$ and $E_{-\vec{\alpha}_a}$ are the generators of the positive and negative simple roots, respectively, and $a, i = 1, ..., n$. The commutation relations for the Chevalley-Serre generators are then

$$
\begin{aligned}
[H_a, H_b] &= 0, \\
[H_a, E_b] &= A_{ab} E_b, \\
[H_a, F_b] &= A_{ab} F_b, \\
[E_a, F_b] &= \delta_{ab} H_a.
\end{aligned}
\tag{B.30}
$$

Taking the Chevalley-Serre generators of a finite dimensional semi-simple Lie algebra $\mathfrak{g}$, with Cartan matrix A, to satisfy the relations

$$[E_a, [E_a, ...E_b] ...] = 0 \tag{B.31}$$

and

$$[F_a, [F_a, ...F_b] ...] = 0 \tag{B.32}$$

one may show that a unique semisimple complex Lie algebra can be constructed.

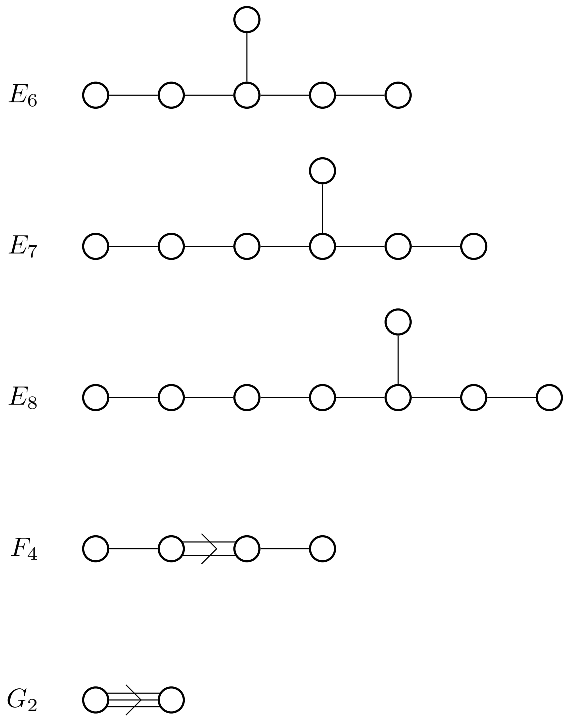

Figure 25: The exceptional Lie algebras

B.2.5 Weights and Representations

Let us now consider how the generators of a rank n semi-simple Lie algebra $\mathfrak{g}$ partitioned into its positive and negative root generators $E_{\vec{\alpha}} \in \mathfrak{g}$ and the Cartan subalgebra $\mathfrak{h}$, act upon a representation space V. Consider an arbitrary irreducible finite dimensional representation of a semi-simple Lie algebra acting on a vector space V. Since the elements of the Cartan subalgebra commute, one may choose a basis in which the states $|\psi\rangle \in V$ belonging to the vector space V are simultaneous eigenvectors of all Cartan subalgebra elements. The action of a Cartan subalgebra element H_i, in Cartan-Weyl basis, on an eigenvector $|\psi_{\vec{\mu}}\rangle \in V$ is given by

$$H_i \, |\psi_{\vec{\mu}}\rangle = \mu_i \, |\psi_{\vec{\mu}}\rangle \tag{B.33}$$

where $H_i \in \mathfrak{h}$ and $\mu_i \in \mathbb{C}$. The complex number μ_i is the i'th component of the **weight vector** $\vec{\mu}$ carried by a state $|\psi_{\vec{\mu}}\rangle$. For a rank n Lie algebra the weight vector $\vec{\mu}$ is an element of an n dimensional vector space. One may define an ordering on the space of weights in the same way we have defined an ordering on the space of roots Δ, by taking a weight vector $\vec{\mu}$ to be positive $\vec{\mu} > 0$ if its first non-zero component $\vec{\mu}_i$ is positive. All weight vectors $\vec{\mu}$ that are not positive or identically zero are then defined to be negative $\vec{\mu} < 0$. From the action of the adjoint representation $(B.10)$ and the commutation relation $(B.20)$, we see that the roots $\vec{\alpha} \in \Delta$ are the weights of the adjoint representation.

The weights are linear maps on the Cartan subalgebra to the complex numbers and are thus elements of the set of linear functionals on the Cartan subalgebra $\mathfrak{h}^*$. It follows that the Killing metric B may be used to define a scalar product on the space of weights $(\,,\,) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$, such that for $\mu, \nu \in \mathfrak{h}^*$,

$$(\mu, \nu) = \mu^i \nu^j g_{ij} = \mu^i \nu_j, \tag{B.34}$$

where g_{ij} is the Killing metric. We now consider how the rest of the algebra $\mathfrak{g}$ acts on the states $|\psi_{\vec{\mu}}\rangle$ of a representation. For a state $|\psi_{\vec{\mu}}\rangle$ of weight $\vec{\mu}$ and a positive root generator $E_{\vec{\alpha}} \in \Delta_+$, one finds

$$\begin{aligned} H_i E_{\vec{\alpha}} |\psi_{\vec{\mu}}\rangle &= [H_i, E_{\vec{\alpha}}] |\psi_{\vec{\mu}}\rangle + E_{\vec{\alpha}} H_i |\psi_{\vec{\mu}}\rangle \\ &= \alpha_i E_{\vec{\alpha}} |\psi_{\vec{\mu}}\rangle + \mu_i E_{\vec{\alpha}} |\psi_{\vec{\mu}}\rangle \\ &= (\mu_i + \alpha_i) E_{\vec{\alpha}} |\psi_{\vec{\mu}}\rangle , \end{aligned} \tag{B.35}$$

where we have made use of equation $(B.20)$. Similarly, a negative root generator $E_{-\vec{\alpha}}$, where $-\vec{\alpha}$ is a negative root, acting on a state $|\psi_{\vec{\mu}}\rangle$ of weight $\vec{\mu}$ gives

$$H_i E_{\vec{\alpha}} |\psi_{\vec{\mu}}\rangle = (\mu_i - \alpha_i) E_{\vec{\alpha}} |\psi_{\vec{\mu}}\rangle . \tag{B.36}$$

So we may view positive root generators as raising operators $E_{\vec{\alpha}} \in \Delta_+$, and negative root generators $E_{\vec{\alpha}} \in \Delta_-$ as lowering operators. We also see that acting on a state $|\psi_{\vec{\mu}}\rangle$ of weight $\vec{\mu}$ with a raising $E_{\vec{\alpha}}$ or lowering operator $E_{-\vec{\alpha}}$ leads to a state $|\psi_{\vec{\mu}\pm\vec{\alpha}}\rangle = E_{\pm\vec{\alpha}} |\psi_{\vec{\mu}}\rangle$ of weight $(\vec{\mu} \pm \vec{\alpha})$. However, for a given finite dimensional representation there is no guarantee that such a state may exist. Instead, taking a state with highest weight $\vec{\Lambda}$ in a finite-dimensional representation one may construct all the states in the representation through the action of the positive and negative root generators as raising and lowering operators. In fact, one only needs the positive root generators $E_{\vec{\alpha}}$ corresponding to the n simple roots $\vec{\alpha}$ and their negative root generator counterparts $E_{-\vec{\alpha}}$. Starting with a highest weight state $|\psi_{\vec{\Lambda}}\rangle$, we have, by definition, that every positive root generator annihilates the highest weight state $|\psi_{\vec{\Lambda}}\rangle$. Acting with a lowering operator $E_{-\vec{\alpha}}$ on the highest weight state $|\psi_{\vec{\Lambda}}\rangle$ it may be shown that a state with weight $\vec{\Lambda} - p\vec{\alpha}$ exists if

$$2\frac{\left(\vec{\alpha}, \vec{\Lambda}\right)}{(\vec{\alpha}, \vec{\alpha})} = p \tag{B.37}$$

where p is a positive integer. In general, a state $|\psi_{\vec{\mu}}\rangle$ with weight $\vec{\mu}$ in the representation will give rise to another state of weight $\vec{\mu} - p\vec{\alpha}$ when acted on by a lowering operator if

$$2\frac{(\vec{\alpha}, \vec{\mu})}{(\vec{\alpha}, \vec{\alpha})} = p, \tag{B.38}$$

where p is a positive integer. Similarly, a state $|\psi_{\vec{\mu}}\rangle$ with weight $\vec{\mu}$ will give rise to another state of weight $\vec{\mu} + q\vec{\alpha}$ when acted on by a raising operator if

$$2\frac{(\vec{\alpha},\vec{\mu})}{(\vec{\alpha},\vec{\alpha})} = -q, \tag{B.39}$$

where q is a positive integer. A representation may then be filled out by starting with a highest weight state $|\psi_{\vec{\Lambda}}\rangle$ and acting on this state, along with all resulting states with the raising and lowering operators until no further states may be constructed. One should note that more than one state in a representation of a Lie algebra may have the same weight. The **multiplicity** of a weight $\vec{\mu}$ is the number of states in a representation of a Lie algebra $\mathfrak{g}$ that possess weight $\vec{\mu}$. All states in representations of the rank n semi-simple Lie algebras that we are concerned with in this thesis have multiplicity one, except the adjoint representation which possesses n states with weight zero, all other states in the adjoint representation have multiplicity one. The **fundamental weight** vectors $\vec{\Lambda}_i$ defined by,

$$2\frac{\left(\vec{\alpha}_i,\vec{\Lambda}_j\right)}{(\vec{\alpha}_i,\vec{\alpha}_i)} = \delta_{ij}, \tag{B.40}$$

where $\vec{\alpha}_i$, $i = 1,...,n$ are the simple roots of the Lie algebra $\mathfrak{g}$, allow us to express the highest weight $\vec{\mu}$ of any representation as

$$\vec{\mu} = \sum_{i=1}^{n} m_i\vec{\Lambda}_i, \tag{B.41}$$

for some set of non-negative integer coefficients m_i. The set of finite dimensional, irreducible representations, of a rank n semi-simple Lie algebra is in one-to-one correpondence with the set of highest weights of the form $\sum_{i=1}^{n} m_i\vec{\Lambda}_i$, with non-negative integer coefficients m_i. Therefore, starting from a highest weight state $\vec{\mu}$, one may construct any finite dimensional, irreducible representation of a semi-simple Lie algebra through the process described above.

Any finite dimensional, irreducible representation of a semi-simple Lie algebra naturally gives rise to a representation of the corresponding semi-simple Lie group through the exponential map defined in equation $(B.7)$. However, to write the group elements of a semi-simple Lie group G in a more useful form we will look to decompose the Lie algebra $\mathfrak{g}$, with positive root generators $E_{\vec{\alpha}}$, negative root generators $E_{-\vec{\alpha}}$ and Cartan subalgebra elements in a particular way. The Lie algebra automorphism $\tau : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\tau : (E_{\vec{\alpha}}, E_{-\vec{\alpha}}, H) \to -(E_{-\vec{\alpha}}, E_{\vec{\alpha}}, H), \tag{B.42}$$

where $\vec{\alpha} > 0$, is known as the **Cartan involution**. Denoting the generators of the subalgebra of $\mathfrak{g}$ preserved by the Cartan involution as $\mathfrak{k}$ we find $\mathfrak{k} = \{(E_{\vec{\alpha}} - E_{-\vec{\alpha}}) \in \mathfrak{g} : \vec{\alpha} \in \Delta^+\}$. The **Iwasawa decomposition** then allows one to write a semisimple Lie algebra $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$, where $\mathfrak{k}$ are defined as above, $\mathfrak{a}$ are the generators of the Cartan subalgebra and $\mathfrak{n}$ are the generators of a nilpotent subalgebra, which we may take to be the positive root generators of $\mathfrak{g}$. Any group element $g \in G$, may then be written $g = KAN$, where K is the maximal compact subgroup of G, generated by $\mathfrak{k}$, A is the maximal abelian subgroup, generated by $\mathfrak{a}$ and N is a nilpotent subgroup generated by $\mathfrak{n}$. In terms of the generators of $\mathfrak{g}$ one can express a representation $D(g)$ of the group element g as

$$D(g) = e^{\sum_{\vec{\alpha}>0} \chi_{\vec{\alpha}} E_{\vec{\alpha}}} e^{-\frac{1}{\sqrt{2}}\vec{\phi}.\vec{H}} e^{\sum_{\vec{\alpha}>0} k_{\vec{\alpha}}(E_{\vec{\alpha}} - E_{-\vec{\alpha}})}, \tag{B.43}$$

where $\chi_{\vec{\alpha}}$, $k_{\vec{\alpha}}$ are the parameters of the positive root generators and the generators of the maximal compact subgroup, respectively, and $\vec{H}$, $\vec{\phi}$ are n vectors of the Cartan subalgebra elements and their parameters, respectively, in the scalar product $\vec{\phi}.\vec{H} = \phi_1 H_1 + ... + \phi_n H_n$. The numerical factor $-\frac{1}{\sqrt{2}}$ in $(B.43)$ is purely a convention adopted in this thesis for convenience and may be absorbed into the definition of the parameters. Therefore, given a finite dimensional, irreducible representation of a semi-simple Lie algebra $\mathfrak{g}$ with generators that are $k \times k$ matrices we may obtain a representation of the corresponding Lie group $\mathfrak{g}$ known as the $\mathbf{k}$ of G through the Iwasawa decomposition, with group elements given by $(B.43)$. Under a transformation by $g \in G$ in the representation $\mathbf{k}$, we have

$$|\psi_{\mathbf{k}}\rangle \to D(g^{-1})\,|\psi_{\mathbf{k}}\rangle \tag{B.44}$$

where $D(g^{-1})$ is the representation of the inverse of the group element g given by the Iwasawa decomposition $(B.43)$ in terms of a $k \times k$ matrix representation of the generators of the Lie algebra $\mathfrak{g}$ and the states $|\psi_{\mathbf{k}}\rangle$ are in a k dimensional vector space V. One may similarly construct the **dual representation** of G, denoted $\overline{\mathbf{k}}$ by taking the states $\langle\psi_{\overline{\mathbf{k}}}| \in V^*$ that are linear functionals on V, which transform as

$$\langle\psi_{\overline{\mathbf{k}}}| \to \langle\psi_{\overline{\mathbf{k}}}|D(g). \tag{B.45}$$

The inner product $\langle\psi_{\overline{\mathbf{k}}}|\psi_{\mathbf{k}}\rangle$ is clearly invariant under G.

In addition to picking out the maximal compact subroup K as the group generated by those Lie algebra elements invariant under the Cartan involution, one may use the Cartan involution τ to define a twisted group representation that will be useful to us when constructing the fields in the effective actions of type IIA/B string theory and M-theory as non-linear representations of the $E_{n+1}(\mathbb{Z})$ U-duality group. If we take the states in the Cartan twisted representation to be $|\psi_{\tau\mathbf{k}}\rangle = \psi_{\tau a}|e^a\rangle$, then the components $\psi_{\tau a}$ transform as

$$\psi_{\tau a} \to U(g)\psi_{\tau a} = D\left(\tau\left(g^{-1}\right)\right)_a{}^b \psi_{\tau b}. \tag{B.46}$$

One may also define a Cartan twisted dual representation, with states $\langle\psi_{\tau\bar{\mathbf{k}}}| = \psi_\tau^a\langle e_a|$. The components of the Cartan twisted dual representation then transform as

$$\psi_\tau^a \to U(g)\psi_{\tau a} = \psi_\tau^b D\left(\tau\left(g\right)\right)_b{}^a. \tag{B.47}$$

B.2.6 $SL(3)$ Example

Let us examine the concepts presented in this section by investigating the Lie algebra of the semi-simple, rank 2, Lie group $SL(3)$. The Cartan matrix A of $SL(3)$ is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{B.48}$$

The $SL(3)$ Dynkin diagram is given in figure 26. The Lie algebra $\mathfrak{sl}(3,\mathbb{R})$ of $SL(3)$ over $\mathbb{R}$ is given

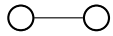

Figure 26: The $SL(3)$ Dynkin diagram

by

$$\mathfrak{sl}(n,\mathbb{R}) = \left\{X \in \mathfrak{gl}(3,\mathbb{R}) : Tr(X) = 0\right\}. \tag{B.49}$$

The dimension of the $SL(3)$ Lie group is 8, so we require 8 basis elements for the $SL(3)$ Lie algebra $\mathfrak{g}$. The Cartan subalgebra will be made up of 2 commuting matrices since $SL(3)$ has rank 2. For the basis of the Cartan subalgebra $\mathfrak{h}$ let us take

$$\begin{aligned} H_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ H_2 &= \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \tag{B.50}$$

Our choice of basis then satisfies $\mathrm{Tr}(H_i H_j) = \delta_{ij}$. We may choose the rest of the $SL(3)$ Lie algebra elements to be

$$\begin{aligned} E_{12} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ F_{12} &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_{13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{B.51}$$

The commutation relations between the Cartan subalgebra elements H_1, H_2 and the positive root generators E_{12}, E_{13}, E_{23} are then

$$\begin{aligned} [a_1 H_1 + a_2 H_2, E_{12}] &= \left(\sqrt{2} a_1\right) E_{12}, \\ [aH_1 + bH_2, E_{13}] &= \left(\frac{1}{\sqrt{2}} a_1 + \sqrt{\frac{3}{2}} a_2\right) E_{13}, \\ [aH_1 + bH_2, E_{12}] &= \left(-\frac{1}{\sqrt{2}} a_1 + \sqrt{\frac{3}{2}} a_2\right) E_{23}. \end{aligned} \tag{B.52}$$

Therefore the positive roots for our choice of basis of the Lie algebra of $SL(3)$ are $\left(\sqrt{2}, 0\right)$, $\left(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$ corresponding to the generators E_{12}, E_{13} and E_{23}, respectively. The negative roots are clearly $\left(-\sqrt{2}, 0\right)$, $\left(-\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}\right)$ and associated with the generators F_{12}, F_{13} and F_{23}, respectively. One may check that our chosen basis with the set of roots found above does indeed satisfy the commutation relations $(B.19 - B.22)$. We then find that the simple roots are

$$\begin{aligned} \vec{\alpha}_1 &= \left(\sqrt{2}, 0\right), \\ \vec{\alpha}_2 &= \left(-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right). \end{aligned} \tag{B.53}$$

The simple roots $\vec{\alpha}_1, \vec{\alpha}_2$ allow us to calculate the fundamental weights that provide a basis for the highest weights of the irreducible representations of $SL(3)$. From equation $(B.40)$, the fundamental weights $\vec{\Lambda}_i$, $i = 1, 2$ for our choice of basis, are given by

$$\begin{aligned} \vec{\Lambda}_1 &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right), \\ \vec{\Lambda}_2 &= \left(0, \sqrt{\frac{2}{3}}\right). \end{aligned} \tag{B.54}$$

It is instructive to calculate the weights and construct the corresponding states in the representation of $\mathfrak{sl}_3$ with highest weight $\vec{\Lambda}_1$. Acting on the state $|\psi_{\vec{\Lambda}_1}\rangle$ with the lowering operator $F_{12} = E_{-\vec{\alpha}_1}$

one finds a state $|\psi_{\vec{\Lambda}_1-\vec{\alpha}_1}\rangle$ since

$$2\frac{\left(\vec{\Lambda}_1,\vec{\alpha}_1\right)}{(\vec{\alpha}_1,\vec{\alpha}_1)}=1, \tag{B.55}$$

acting on the highest weight state $|\psi_{\vec{\Lambda}_1}\rangle$ with the lowering operator $F_{23}=E_{-\vec{\alpha}_2}$ annihilates the state highest weight state, so the only state that may be constructed from the highest weight state through the action of a single lowering operator is $|\psi_{\vec{\Lambda}_1-\vec{\alpha}_1}\rangle$. Acting on the state $|\psi_{\vec{\Lambda}_1-\vec{\alpha}_1}\rangle$ with the lowering operator $E_{-\vec{\alpha}_1}$ annihilates the state, since

$$2\frac{\left(\vec{\Lambda}_1,\vec{\alpha}_1\right)}{(\vec{\alpha}_1,\vec{\alpha}_1)}=-1, \tag{B.56}$$

while acting with the lowering operator $E_{-\vec{\alpha}_2}$ gives a state $|\psi_{\vec{\Lambda}_1-\vec{\alpha}_1-\vec{\alpha}_2}\rangle=E_{-\vec{\alpha}_2}|\psi_{\vec{\Lambda}_1-\vec{\alpha}_1}\rangle$ as

$$2\frac{\left(\vec{\Lambda}_1,\vec{\alpha}_1\right)}{(\vec{\alpha}_1,\vec{\alpha}_1)}=1. \tag{B.57}$$

We then find that both lowering operators $E_{-\vec{\alpha}_1}$ and $E_{-\vec{\alpha}_2}$ annihilate the state $|\psi_{\vec{\Lambda}_1-\vec{\alpha}_1-\vec{\alpha}_2}\rangle$ so all the states in the representation of $\mathfrak{sl}_3$ with highest weight $\vec{\Lambda}_1$ have been found. This representation of $\mathfrak{sl}_3$ is known as the **3** of $SL(3)$, one may check that taking

$$|\psi_{\vec{\Lambda}_1}\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix},\ |\psi_{\vec{\Lambda}_1-\vec{\alpha}_1}\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix},\ |\psi_{\vec{\Lambda}_1-\vec{\alpha}_1-\vec{\alpha}_2}\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{B.58}$$

leads to the expected weights under the action of the Cartan subalgebra $\mathfrak{h}$.

B.3 Non-linear Realisations

It is well known that the scalar fields in the $d=10-n$ dimensional maximal supergravity theories possess a $E_{n+1}(\mathbb{R})$ global symmetry and a local symmetry given by the maximal compact group H. The scalars in a $d=10-n$ dimensional maximal supergravity theory parametrise the coset $E_{n+1}(\mathbb{R})/H(\mathbb{R})$ and transform in the adjoint representation of $E_n(\mathbb{R})$, while the gauge fields lie in various representations of $E_{n+1}(\mathbb{R})$ and transform linearly under a $E_{n+1}(\mathbb{R})$ group transformation. Terms in the $d=10-n$ dimensional maximal supergravity action are then constructed from these linear representations of the group G and are invariant under this symmetry. However, it will be desirable for us to use the theory of non-linear realisations to write the $d=10-n$ dimensional maximal supergravity action in terms of fields transforming under non-linear representations of $E_{n+1}(\mathbb{R})$ and the Cartan forms of $E_{n+1}(\mathbb{R})/H(\mathbb{R})$.

Consider a group G with subgroup H. For $g \in G$, a natural action on the group G is given by

$$g \to g_0 g, \tag{B.59}$$

where $g_0 \in G$. However, if one considers the coset space constructed by taking G/H and the natural action on the coset space G/H through the group G in equation (B.59), then we find that this action does not in general preserve a given coset representative $g \in G/H$. The preservation of a choice of coset representative $g \in G/H$ requires a compensating transformation which is provided by the action of the subgroup H on g. Elements of the coset space then transform as

$$g \to g_0 g h^{-1}, \tag{B.60}$$

where $h^{-1} \in H$. In this thesis we are interested in taking a group G, with spacetime dependent group elements $g \in G$ and a local subgroup H, where the group elements $g \in G$ transform as

$$g \to g_0 g h^{-1}, \tag{B.61}$$

where $h \in H$ is also spacetime dependent and $g_0 \in G$ is a constant group element, this is known as a **non-linear realization** of G with respect to H. If we denote the coset representatives by $g(\xi)$, where ξ labels the equivalence classes in the coset, then under a g_0 transformation we have

$$g(\xi) \to g_0 g(\xi) = g'(g_0 \cdot \xi) h(g_0, \xi). \tag{B.62}$$

The right multiplication of the coset element $g' \in G/H$ by the subgroup element $h(g_0, \xi)$ is the compensating transformation required to preserve our choice of coset representative. For the semisimple Lie groups $E_{n+1}(\mathbb{R})$, we may make use of the Iwasawa decomposition (B.43) and the local symmetry H, to write the coset element $g \in E_{n+1}/H$ in terms of the generators of a representation of the $E_{n+1}(\mathbb{R})$ Lie algebra $\mathfrak{g}$ in Cartan-Weyl basis, as

$$D(g) = e^{\sum_{\vec{\alpha}>0} \chi_{\vec{\alpha}} E_{\vec{\alpha}}} e^{-\frac{1}{\sqrt{2}} \vec{\phi}.\vec{H}}, \tag{B.63}$$

where $\chi_{\vec{\alpha}}$ and $\vec{\phi}$ are the fields parameterising the coset. A representation of the coset $g \in E_{n+1}/H$ is the basic object from which the Cartan forms are constructed to give the derivatives of the scalars appearing in the effective action of type IIA/B string theory and M-theory in $d = 10 - n$ dimensions and the object used to convert the field strengths formed from the gauge fields that carry a linear representation of $E_{n+1}(\mathbb{R})$ to a non-linear representation.

B.3.1 Cartan Forms

For a Lie group G, the Lie algebra valued one-form gdg^{-1}, where $g \in G$ and d is the exterior derivative, is the **Cartan form** of G. We will be interested in the case where the Lie group is the quotient group E_{n+1}/H, as such, the coset element $g \in E_{n+1}/H$ transforms as $g \to g_0 gh$ where $g_0 \in E_{n+1}$ is a rigid group element and $h \in H$ is a local element of the maximal compact subgroup H. Denoting the Cartan form of E_{n+1}/H by $\mathcal{V} = gdg^{-1}$, we find $\mathcal{V}$ transforms as

$$\begin{aligned}\mathcal{V} &= g^{-1}dg \\ &\to (g_0 gh)^{-1} d (g_0 gh) \\ &= h^{-1}g^{-1}g_0^{-1}g_0 d(gh) \\ &= h^{-1}g^{-1}((dg)h + g(dh)) \\ &= h^{-1}\mathcal{V}h + h^{-1}dh.\end{aligned} \tag{B.64}$$

Defining the generalised transpose $^{\#} : \mathfrak{g} \to \mathfrak{g}$ through the Cartan involution as $A^{\#} = -\tau(A)$, one may split the Cartan form $\mathcal{V}$ into its symmetric and antisymmetric parts under the action of the generalised transpose $\#$, by taking

$$\mathcal{V} = \mathcal{S} + \omega, \tag{B.65}$$

where the symmetric $\mathcal{S}$ and antisymmetric ω parts of $\mathcal{V}$, given in terms of the Cartan involution τ, are

$$\begin{aligned}\mathcal{S} &= \frac{1}{2}(\mathcal{V} - \tau(\mathcal{V})), \\ \omega &= \frac{1}{2}(\mathcal{V} + \tau(\mathcal{V})).\end{aligned} \tag{B.66}$$

The symmetric part $\mathcal{S}$ of the Cartan form $\mathcal{V}$, transforms as

$$\begin{aligned}\mathcal{S} \to&\frac{1}{2}\left(h^{-1}\mathcal{V}h + h^{-1}dh - \tau\left(h^{-1}\mathcal{V}h + h^{-1}dh\right)\right) \\ =&\frac{1}{2}\left(h^{-1}\mathcal{V}h + h^{-1}dh - h^{-1}\tau(\mathcal{V})h + h^{-1}dh\right) \\ =&\frac{1}{2}\left(h^{-1}\mathcal{V}h - h^{-1}\tau(\mathcal{V})h\right) \\ =&h^{-1}\mathcal{S}h,\end{aligned} \tag{B.67}$$

while the antisymmetric part ω of the Cartan form $\mathcal{V}$, transforms as

$$\begin{aligned}\omega \to&\frac{1}{2}\left(\left(h^{-1}\mathcal{V}h + h^{-1}dh\right) + \tau\left(h^{-1}\mathcal{V}h + h^{-1}dh\right)\right) \\ =&\frac{1}{2}h^{-1}\mathcal{V}h + \frac{1}{2}h^{-1}dh + \frac{1}{2}\left(\tau\left(h^{-1}\right)\tau(\mathcal{V})\tau(h) + \tau\left(h^{-1}\right)\tau(dh)\right) \\ =&\frac{1}{2}h^{-1}\mathcal{V}h + \frac{1}{2}h^{-1}dh + \frac{1}{2}h^{-1}\tau(\mathcal{V})h + \frac{1}{2}h^{-1}dh \\ =&h^{-1}\mathcal{S}h + h^{-1}dh.\end{aligned} \tag{B.68}$$

Additionally, under the Cartan involution τ we have

$$\begin{aligned}\tau(\mathcal{S}) =& \frac{1}{2}\left(\tau(\mathcal{V}) - \tau(\tau(\mathcal{V}))\right)\\ =& \frac{1}{2}\left(-\mathcal{V} + \tau(\mathcal{V})\right)\\ =& -\mathcal{S}\end{aligned} \tag{B.69}$$

and

$$\begin{aligned}\tau(\omega) =& \frac{1}{2}\left(\tau(\mathcal{V}) + \tau(\tau(\mathcal{V}))\right)\\ =& \frac{1}{2}\left(\mathcal{V} + \tau(\mathcal{V})\right)\\ =& \omega\end{aligned} \tag{B.70}$$

thus ω lies in the subalgebra $\mathfrak{k}$ while $\mathcal{S}$ lies in the complement $\mathfrak{g} - \mathfrak{k}$, where $\mathfrak{k}$ is the subalgebra preserved by the Cartan involution that generates the maximal compact subgroup H of G. The dynamics of the scalars appearing in the effective action of type IIA/B string theory and M-theory in $d = 10 - n$ dimensions may then be constructed by tracing over the group indices of products of the symmetric part $\mathcal{S}_\mu$ of the Cartan form $\mathcal{V}_\mu$ for the group E_{n+1}/H. In particular, the two derivative scalar terms in the Einstein frame action may be written

$$S_{Scalar} = \frac{1}{2\kappa_d}\int d^dx \left(\mathcal{S}_{\mu\bar{I}}{}^{\bar{J}}\mathcal{S}^{\mu\bar{I}}_{\bar{J}}\right), \tag{B.71}$$

where the overlined upper case Roman indices are internal H indices.

B.3.2 Non-linear Representations

Consider a linear representation of G with states $|\psi\rangle = \sum_{a=1}^{N}\psi_a\,|\psi_{\vec{\mu}^a}\rangle$ belonging to an N dimenional vector space V, where we have expanded the state $|\psi\rangle$ in a complete set of eigenvectors $|\psi_{\vec{\mu}^i}\rangle$ of the Cartan subalgebra, with corresponding components ψ_a. The components ψ_a transform under a rigid group element $g_0 \in G$ as

$$U(g_0)\psi_b = D\left({g_0}^{-1}\right)_a{}^b\,\psi_b. \tag{B.72}$$

By taking a coset element $g^{-1} \in G/H$, where H is a subgroup of G and ξ are the coordinates of the coset element g^{-1}, one may convert a linear representation of G to a non-linear representation by defining

$$\varphi_a(\xi) = D((g^{-1}(\xi))_a{}^b\psi_b. \tag{B.73}$$

Elements of the non-linear representation $\varphi(\xi)$ then transform as

$$\begin{aligned}U(g_0)\varphi_a(\xi) =& D((g^{-1}(\xi))_a{}^b D(g_0{}^{-1})b \quad {}^c\psi_c \\ =& D((g_0 g(\xi))^{-1})_a{}^b\psi_b \\ =& D((g(\xi')h(g_0,\xi))^{-1})_a{}^b\psi_b \\ =& D(h^{-1}(g_0,\xi))_a{}^b\varphi(\xi')_b.\end{aligned} \tag{B.74}$$

So under the action of G, φ transforms by some matrix representation of $h(g_0, \xi) \in H$, which is a function of the rigid group element g_0 and the coset coordinate ξ. In general the compensating transformation $D(h^{-1}(g_0, \xi))$ may always be found although solving explicitly for the components $D(h^{-1}(g_0, \xi))_a{}^b$ may be difficult.

We may also convert any linear N dimensional dual representation of G acting on representation space elements $\langle\psi| = \psi^a\langle\psi_{\vec{\mu}_a}|$, with components ψ^a that transform as

$$U(g_0)\psi^a = \psi^b D\left(g_0\right)_b{}^a, \tag{B.75}$$

to a non-linear representation of G with respect to H by defining

$$\varphi_D^a(\xi) = \psi^b D((g(\xi))_b{}^a. \tag{B.76}$$

Elements of the non-linear representation $\varphi(\xi)$ then transform as

$$U(g_0)\varphi_D^a(\xi) = \varphi(\xi')^b D(h(g_0,\xi))_b{}^a. \tag{B.77}$$

One could then construct invariants under the action of G by taking terms of the form $\varphi_D^a(\xi)\varphi_a(\xi)$, however these terms are trivially invariant and do not give terms that we may identify as the field strength bilinears that appear in the effective action of type IIA/B string theory and M-theory in $d = 10 - n$ dimensions. Instead, one must convert the Cartan twisted and Cartan twisted dual representations to non-linear representations of the symmetry group G with respect to its maximal compact subgroup H by defining

$$\varphi_{\tau a} = D\left(g^{\#}\right)_a{}^b\psi_{\tau b} \tag{B.78}$$

and

$$\varphi_\tau^a = \psi^{\tau b} D\left(\left(g^{\#}\right)^{-1}\right)_b{}^a, \tag{B.79}$$

where $g^{\#} = (\tau(g))^{-1}$. The components of the Cartan twisted non-linear representation $\varphi_{\tau a}$ and the Cartan twisted dual non-linear representation φ_τ^a then transform as

$$U(g_0)\varphi_{\tau a}(\xi) = D\left(h\left(g_0,\xi\right)^{-1}\right)_a{}^b\psi_{\tau b} \tag{B.80}$$

and

$$U(g_0)\varphi_\tau^a\left(\xi\right) = \psi^{\tau b} D\left(h\left(g_0, \xi\right)\right)_b{}^a. \tag{B.81}$$

So we see that the term

$$\begin{aligned}\varphi_\tau^a\left(\xi\right)\varphi_a\left(\xi\right) &= \psi^{\tau b} D\left(\left(g^{\#}\left(\xi\right)\right)^{-1}\right)_b{}^a D((g^{-1}(\xi))_b{}^c\psi_c \\ &= \psi^{\tau b} D(M^{-1})_b{}^c\psi_c,\end{aligned} \tag{B.82}$$

where $M = gg^{\#}$, is invariant under rigid transformations $g_0 \in G$. The Cartan twisted dual representation with components ψ_τ^a and the representation with components ψ_a are isomorphic and the non-linear representations constructed from them transform under the maximal compact subgroup H, therefore the term $\varphi_\tau^a\left(\xi\right)\varphi_a\left(\xi\right)$ may be written $\varphi_{\overline{I}}\varphi_{\overline{J}}\delta^{\overline{IJ}}$, where overlined upper case Roman indices are internal H indices. Since the non-linear representations of G transform under the maximal compact subgroup H it is possible to construct invariants through contracting H indices of covariant or invariant H tensors. For instance the maximal compact subgroup of $SL(n)$ is $SO(n)$, therefore if $\varphi_{\overline{I}}$ a non-linear representation of $SL(n)$ with respect to $SO(n)$ one may construct an invariant by taking $\epsilon^{\overline{I}_1...\overline{I}_n}\varphi_{\overline{I}_1}...\varphi_{\overline{I}_n}$, where $\epsilon^{\overline{I}_1...\overline{I}_n}$ is the totally antisymmetric tensor, which is an $SO(n)$ invariant. The term $\varphi^a(\xi)\varphi_{\tau a}(\xi)$ is similarly invariant under G, however it is related to $\varphi_\tau^a\left(\xi\right)\varphi_a\left(\xi\right)$ by a change of basis of the Lie algebra $\mathfrak{g}$, therefore we find no new invariants from considering terms of this form.

C Evaluation of the Perturbative Parts of Unconstrained Eisenstein-like Automorphic Forms

In this section we derive the perturbative parts of the unconstrained Eisentein-like automorphic forms contructed from representations of E_{n+1} considered in chapter 8. The relevant weight information needed to construct the perturbative parts of the automorphic forms are given in table form for each representation, where j_n and j_{n+1} are equal to the absolute value of the coefficients of the $\vec{\alpha}_n$ and $\vec{\alpha}_{n+1}$ simple roots in each weight $\vec{\mu}_i$ in the root string. One may then use (8.43) to write down the perturbative part of the automorphic form. Note that the ordering of the weights in the representation is such that $\vec{\mu}_i > \vec{\mu}_j$ if the first non-zero component of $\vec{\mu}_i - \vec{\mu}_j$ in the ordered basis $(0,1,\underline{0})$, $(1,0,\underline{0})$ and $(0,0,\underline{\mu})$ is positive.

C.1 10 of $SL(5)$

The $SL(2) \times SL(n)$ decomposition of the representation of $SL(5)$ with highest weight $\vec{\Lambda}^1$ is given in table 11.

The perturbative part of the unconstrained Eisenstein-like automorphic form constructed from the

$(j_n,\, j_{n+1})$	(0,0)	(1,1)	(1,2)	(2,0)	
$SL(m)$ rep.	$\underline{\lambda}^1$	$\underline{\lambda}^2$	$\underline{\lambda}^2$	$\underline{0}$	
$SL(2)$ weight	0	μ	$-\mu$	0	
d_α	3	3	3	1	
a_α	0	3	6	9	10
b_α	0	0	3	6	8

Table 11: Decomposition of the **10** of $SL(5)$

representation of $SL(5)$ with highest weight $\vec{\Lambda}^1$ is given by

$$\begin{aligned}
\Phi_p =& \sum_{k=1}^{3} e^{-\frac{2\sqrt{2}s}{3x}\rho} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=4}^{6} e^{-\frac{2\sqrt{2}s}{3x}\rho} e^{(2s-3)\frac{x}{\sqrt{2}}\rho} e^{-\phi\left(s-\frac{3}{2}\right)} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) \\
&+ \sum_{k=7}^{9} e^{-\frac{2\sqrt{2}s}{3x}\rho} e^{(2s-3)\frac{x}{\sqrt{2}}\rho} e^{\phi\left(s-\frac{9}{2}\right)} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) + e^{-\frac{2\sqrt{2}s}{3x}\rho} e^{(4s-12)\frac{x}{\sqrt{2}}\rho} E_{10} \\
=& \sum_{k=1}^{3} V_n^{\frac{2s}{3}} g_d^{-\frac{4}{5}s} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=4}^{6} V_n^{\frac{2s}{3}} g_d^{-\frac{4}{5}s} g_d^{\frac{2s-3}{2}} V_n^{-\frac{5}{12}(2s-3)} V_n^{-\frac{s-\frac{3}{2}}{2}} g_d^{-\left(s-\frac{3}{2}\right)} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) \\
&+ \sum_{k=7}^{9} V_n^{\frac{2s}{3}} g_d^{-\frac{4}{5}s} g_d^{\frac{2s-3}{2}} V_n^{-\frac{5}{12}(2s-3)} V_n^{\frac{s-\frac{9}{2}}{2}} g_d^{\left(s-\frac{9}{2}\right)} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) + V_n^{\frac{2s}{3}} g_d^{-\frac{4}{5}s} E_{10} \\
=& \sum_{k=1}^{3} V_n^{\frac{2s}{3}} g_d^{-\frac{4}{5}s} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=4}^{6} V_n^{-\frac{2}{3}s+2} g_d^{-\frac{4}{5}s} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) \\
&+ \sum_{k=7}^{9} V_n^{\frac{1}{3}s-1} g_d^{\frac{6}{5}s-6} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) + V_n^{-s+5} g_d^{\frac{6}{5}s-6} E_{10}.
\end{aligned} \tag{C.1}$$

C.2 16 of $SO(5,5)$

The $SL(2) \times SL(n)$ decomposition of the representation of $SL(5)$ with highest weight $\vec{\Lambda}^1$ is given in table 12.

$(j_n,\, j_{n+1})$	(0,0)	(1,1)	(1,2)	(2,0)	
$SL(m)$ rep.	$\underline{\lambda}^1$	$\underline{\lambda}^3$	$\underline{\lambda}^3$	$\underline{\lambda}^1$	
$SL(2)$ weight	0	μ	$-\mu$	0	
d_α	4	4	4	4	
a_α	0	4	8	12	16
b_α	0	0	4	8	16
$n_c a_\alpha - b_\alpha$	0	4	4	16	

Table 12: Decomposition of the **16** of $SO(5,5)$

The perturbative part of the unconstrained Eisenstein-like automorphic form constructed from the representation of $SO(5,5)$ with highest weight $\vec{\Lambda}^1$ is given by

$$\begin{aligned}
\phi_p &= \sum_{k=1}^{4} e^{-\frac{s}{\sqrt{2}x}\rho} E_k P_k \left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=5}^{8} e^{-\frac{s}{\sqrt{2}x}\rho} e^{(2s-4)\frac{x}{\sqrt{2}}\rho} e^{-\phi(s-2)} E_k P_k \left(\underline{\lambda}_3, \underline{\phi}\right) \\
&+ \sum_{k=9}^{12} e^{-\frac{s}{\sqrt{2}x}\rho} e^{(2s-4)\frac{x}{\sqrt{2}}\rho} e^{\phi(s-6)} E_k P_k \left(\underline{\lambda}_3, \underline{\phi}\right) + \sum_{k=13}^{16} e^{-\frac{s}{\sqrt{2}x}\rho} e^{(4s-16)\frac{x}{\sqrt{2}}\rho} E_k P_k \left(\underline{\lambda}_1, \underline{\phi}\right) \\
&= \sum_{k=1}^{4} V_n^{\frac{s}{2}} g_d^{-s} E_k P_k \left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=5}^{8} V_n^{\frac{s}{2}} g_d^{-s} g_d^{\frac{2s-4}{2}} V_n^{-\frac{(2s-4)}{4}} V_n^{-\frac{s-2}{2}} g_d^{-(s-2)} E_k P_k \left(\underline{\lambda}_3, \underline{\phi}\right) \\
&+ \sum_{k=9}^{12} V_n^{\frac{s}{2}} g_d^{-s} g_d^{\frac{2s-4}{2}} V_n^{-\frac{(2s-4)}{4}} V_n^{\frac{s-6}{2}} g_d^{(s-6)} E_k P_k \left(\underline{\lambda}_3, \underline{\phi}\right) + \sum_{k=13}^{16} V_n^{\frac{s}{2}} g_d^{-s} g_d^{\frac{4s-16}{2}} V_n^{-\frac{(4s-16)}{4}} E_k P_k \left(\underline{\lambda}_1, \underline{\phi}\right)
\end{aligned} \tag{C.2}$$

C.3 24 of $SL(5)$

The $SL(2) \times SL(n)$ decomposition of the representation of $SL(5)$ with highest weight $\vec{\Lambda}^{n-1} + \vec{\Lambda}^{n+1}$ is given in table 13.

(j_n, j_{n+1})	(0,1)	(0,2)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
$SL(m)$ rep.	$\underline{\lambda}^2$	$\underline{\lambda}^2$	$\underline{0}$	$\left(\underline{\lambda}^1 + \underline{\lambda}^2\right)^+$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$SL(2)$ weight	μ	$-\mu$	2μ	0	0	0	0	0
d_α	3	3	1	3	1	1	1	1
a_α	0	3	6	7	10	11	12	13
b_α	0	0	0	1	4	5	6	7
$n_c a_\alpha - b_\alpha$	0	0	6	6	6	6	6	6

	(1,7)	(1,7)	(2,1)	(2,2)	
$SL(m)$ rep.	$\left(\underline{\lambda}^1 + \underline{\lambda}^2\right)^-$	$\underline{0}$	$\underline{\lambda}^1$	$\underline{\lambda}^1$	
$SL(2)$ weight	0	-2μ	μ	$-\mu$	
d_α	3	1	3	3	
a_α	14	17	18	21	24
b_α	8	11	12	18	24
$n_c a_\alpha - b_\alpha$	6	6	6	24	24

Table 13: Decomposition of the **24** of $SL(5)$

The perturbative part of the unconstrained Eisenstein-like automorphic form constructed from the representation of **24** of $SL(5)$ with highest weight $\vec{\Lambda}^1 + \vec{\Lambda}^4$ is given by

$$
\begin{aligned}
\Phi_p = & \sum_{k=1}^{3} e^{-\frac{5s}{3\sqrt{2}x}\rho} e^{-\phi s} E_k P_k\left(\underline{\lambda}_2, \phi\right) + \sum_{k=4}^{6} e^{-\frac{5s}{3\sqrt{2}x}\rho} e^{\phi(s-3)} E_k P_k\left(\underline{\lambda}_2, \phi\right) \\
& + e^{-\frac{5s}{3\sqrt{2}x}\rho} e^{(2s-6)\frac{x}{\sqrt{2}}\rho} e^{-\phi(2s-6)} E_7 + \sum_{k=8}^{10} e^{-\frac{5s}{3\sqrt{2}x}\rho} e^{(2s-6)\frac{x}{\sqrt{2}}\rho} E_k P_k\left(\left(\underline{\lambda}_1 + \underline{\lambda}_2\right)^+, \phi\right) \\
& + e^{-\frac{5s}{3\sqrt{2}x}\rho} e^{(2s-6)\frac{x}{\sqrt{2}}\rho} e^{\phi} E_{11} + e^{-\frac{5s}{3\sqrt{2}x}\rho} e^{(2s-6)\frac{x}{\sqrt{2}}\rho} E_{12} \\
& + e^{-\frac{5s}{3\sqrt{2}x}\rho} e^{(2s-6)\frac{x}{\sqrt{2}}\rho} E_{13} + e^{-\frac{5s}{3\sqrt{2}x}\rho} e^{(2s-6)\frac{x}{\sqrt{2}}\rho} E_{14} \\
& + \sum_{k=15}^{17} e^{-\frac{5s}{3\sqrt{2}x}\rho} e^{(2s-6)\frac{x}{\sqrt{2}}\rho} E_k P_k\left(\left(\underline{\lambda}_1 + \underline{\lambda}_2\right)^-, \phi\right) \\
& + e^{-\frac{5s}{3\sqrt{2}x}\rho} e^{(2s-6)\frac{x}{\sqrt{2}}\rho} e^{\phi(2s-18)} E_{18} \\
& + \sum_{k=19}^{21} e^{-\frac{5s}{3\sqrt{2}x}\rho} e^{(4s-24)\frac{x}{\sqrt{2}}\rho} e^{-\phi(s-9)} E_k P_k\left(\underline{\lambda}_1, \phi\right) \\
& + \sum_{k=22}^{24} e^{-\frac{5s}{3\sqrt{2}x}\rho} e^{(4s-24)\frac{x}{\sqrt{2}}\rho} e^{\phi(s-12)} E_k P_k\left(\underline{\lambda}_1, \phi\right)
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{3} V_n^{\frac{5}{6}s} g_d^{-s} g_d^{-s} V_n^{-\frac{s}{2}} E_k P_k\left(\underline{\lambda}_2\right) + \sum_{k=4}^{6} V_n^{\frac{5}{6}s} g_d^{-s} g_d^{s-3} V_n^{\frac{s-3}{2}} E_k P_k\left(\underline{\lambda}_2\right) \\
&\quad + V_n^{\frac{5}{6}s} g_d^{-s} g_d^{\frac{2s-6}{2}} V_n^{-\frac{5}{12}(2s-6)} g_d^{-(2s-6)} V_n^{-\frac{(2s-6)}{2}} E_7 \\
&\quad + \sum_{k=8}^{10} V_n^{\frac{5}{6}s} g_d^{-s} g_d^{\frac{2s-6}{2}} V_n^{-\frac{5}{12}(2s-6)} g_d^{-1} V_n^{-\frac{1}{2}} E_k P_k\left(\left(\underline{\lambda}_1 + \underline{\lambda}_2\right)^+, \underline{\phi}\right) \\
&\quad + V_n^{\frac{5}{6}s} g_d^{-s} g_d^{\frac{2s-6}{2}} V_n^{-\frac{5}{12}(2s-6)} g_d^{-1} V_n^{-\frac{1}{2}} E_{11} + V_n^{\frac{5}{6}s} g_d^{-s} g_d^{\frac{2s-6}{2}} V_n^{-\frac{5}{12}(2s-6)} g_d^{-1} V_n^{-\frac{1}{2}} E_{12} \\
&\quad + V_n^{\frac{5}{6}s} g_d^{-s} g_d^{\frac{2s-6}{2}} V_n^{-\frac{5}{12}(2s-6)} g_d^{-1} V_n^{-\frac{1}{2}} E_{13} + V_n^{\frac{5}{6}s} g_d^{-s} g_d^{\frac{2s-6}{2}} V_n^{-\frac{5}{12}(2s-6)} g_d^{-1} V_n^{-\frac{1}{2}} E_{14} \\
&\quad + \sum_{k=15}^{17} V_n^{\frac{5}{6}s} g_d^{-s} g_d^{\frac{2s-6}{2}} V_n^{-\frac{5}{12}(2s-6)} g_d^{-1} V_n^{-\frac{1}{2}} E_k P_k\left(\left(\underline{\lambda}_1 + \underline{\lambda}_2\right)^-, \underline{\phi}\right) \\
&\quad + V_n^{\frac{5}{6}s} g_d^{-s} g_d^{\frac{2s-6}{2}} V_n^{-\frac{5}{12}(2s-6)} g_d^{(2s-18)} V_n^{\frac{(2s-18)}{2}} E_{18} \\
&\quad + \sum_{k=19}^{21} V_n^{\frac{5}{6}s} g_d^{-s} g_d^{\frac{4s-24}{2}} V_n^{-\frac{5}{12}(4s-24)} g_d^{-(s-9)} V_n^{-\frac{(s-9)}{2}} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) \\
&\quad + \sum_{k=22}^{24} V_n^{\frac{5}{6}s} g_d^{-s} g_d^{\frac{4s-24}{2}} V_n^{-\frac{5}{12}(4s-24)} g_d^{(s-12)} V_n^{\frac{(s-12)}{2}} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) \\
&= \sum_{k=1}^{3} g_d^{-2s} V_n^{\frac{1}{3}s} E_k P_k\left(\underline{\lambda}_2, \phi\right) + \sum_{k=4}^{6} V_n^{\frac{4}{3}s-\frac{3}{2}} g_d^{-3} E_k P_k\left(\underline{\lambda}_2, \phi\right) \\
&\quad + V_n^{-s+\frac{11}{2}} g_d^{-2s+3} E_7 + \sum_{k=8}^{10} V_n^2 g_d^{-4} E_k P_k\left(\left(\underline{\lambda}_1 + \underline{\lambda}_2, \phi\right)\right)^+ \\
&\quad + V_n^2 g_d^{-4} E_{11} + V_n^2 g_d^{-4} E_{12} \\
&\quad + V_n^2 g_d^{-4} E_{13} + V_n^2 g_d^{-4} E_{14} \\
&\quad + \sum_{k=15}^{17} V_n^2 g_d^{-4} E_k P_k\left(\left(\underline{\lambda}_1 + \underline{\lambda}_2, \phi\right)\right)^- \\
&\quad + V_n^{s-\frac{13}{2}} g_d^{2s-21} E_{18} \\
&\quad + \sum_{k=19}^{21} V_n^{-\frac{4}{3}s+\frac{29}{2}} g_d^{-3} E_k P_k\left(\underline{\lambda}_1, \phi\right) \\
&\quad + \sum_{k=22}^{24} V_n^{-\frac{1}{3}s+4} g_d^{2s-24} E_k P_k\left(\underline{\lambda}_1, \phi\right).
\end{aligned}
\tag{C.4}
$$

C.4 78 of E_6

The $SL(2) \times SL(n)$ decomposition of the representation of E_6 with highest weight $\vec{\Lambda}^{n-1}$ is given in table 14. The perturbative part of the unconstrained Eisenstein-like automorphic form constructed

$(j_n,\, j_{n+1})$	(0,1)	(1,1)	(1,2)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
$SL(m)$ rep.	$\underline{\lambda}^4$	$\underline{\lambda}^2$	$\underline{\lambda}^2$	$\underline{0}$	$(\underline{\lambda}_1+\underline{\lambda}_4)^+$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$SL(2)$ weight	0	μ	$-\mu$	2μ	0	0	0	0	0
d_α	5	10	10	1	10	1	1	1	1
a_α	0	5	15	25	26	36	37	38	39
b_α	0	0	10	20	22	42	44	46	48
$n_c a_\alpha - b_\alpha$	0	5	5	30	30	30	30	30	30

$(j_n,\, j_{n+1})$	(2,7)	(2,8)	(2,9)	(2,10)	(3,1)	(3,2)	(4,0)	
$SL(m)$ rep.	$\underline{0}$	$\underline{0}$	$(\underline{\lambda}_1+\underline{\lambda}_4)^-$	$\underline{0}$	$\underline{\lambda}_3$	$\underline{\lambda}_3$	$\underline{\lambda}_1$	
$SL(2)$ weight	0	0	0	-2μ	μ	$-\mu$	0	
d_α	1	1	10	1	10	10	5	
a_α	40	41	42	52	53	63	73	78
b_α	50	52	54	74	76	106	136	156
$n_c a_\alpha - b_\alpha$	30	30	30	30	83	83	156	

Table 14: Decomposition of the **78** of E_6

from the representation of E_6 with highest weight $\Lambda^{\overrightarrow{n-1}}$ is given by

$$\begin{aligned}
\Phi_p =& \sum_{k=1}^{5} e^{-\frac{6s}{5\sqrt{2x}}\rho} E_k P_k\left(\underline{\lambda}^4,\phi\right) + \sum_{k=6}^{15} e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(2s-5)\frac{x}{\sqrt{2}}\rho} e^{-\phi\left(s-\frac{5}{2}\right)} E_k P_k\left(\underline{\lambda}^2,\underline{\phi}\right) \\
&+ \sum_{k=16}^{25} e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(2s-5)\frac{x}{\sqrt{2}}\rho} e^{\phi\left(s-\frac{25}{2}\right)} E_k P_k\left(\underline{\lambda}^2,\phi\right) + e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(4s-30)\frac{x}{\sqrt{2}}\rho} e^{-\phi(2s-25)} E_{26} \\
&+ \sum_{k=27}^{36} e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(4s-30)\frac{x}{\sqrt{2}}\rho} e^{-\phi} E_k P_k\left(\left(\underline{\lambda}^1+\underline{\lambda}^4\right)^+,\underline{\phi}\right) \\
&+ e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(4s-30)\frac{x}{\sqrt{2}}\rho} e^{-\phi} E_{37} + e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(4s-30)\frac{x}{\sqrt{2}}\rho} E_{38} \\
&+ e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(4s-30)\frac{x}{\sqrt{2}}\rho} e^{-\phi} E_{39} + e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(4s-30)\frac{x}{\sqrt{2}}\rho} E_{40} \\
&+ e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(4s-30)\frac{x}{\sqrt{2}}\rho} e^{-\phi} E_{41} + e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(4s-30)\frac{x}{\sqrt{2}}\rho} E_{42} \\
&+ \sum_{k=43}^{52} e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(4s-30)\frac{x}{\sqrt{2}}\rho} e^{-\phi} E_k P_k\left(\left(\underline{\lambda}^1+\underline{\lambda}^4\right)^-,\underline{\phi}\right) + e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(4s-30)\frac{x}{\sqrt{2}}\rho} e^{\phi(2s-53)} E_{53} \\
&+ \sum_{k=54}^{63} e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(6s-83)\frac{x}{\sqrt{2}}\rho} e^{-\phi\left(s-\frac{53}{2}\right)} E_k P_k\left(\underline{\lambda}^3,\underline{\phi}\right) \\
&+ \sum_{k=64}^{73} e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(6s-83)\frac{x}{\sqrt{2}}\rho} e^{\phi\left(s-\frac{73}{2}\right)} E_k P_k\left(\underline{\lambda}^3,\underline{\phi}\right) \\
&+ \sum_{k=74}^{78} e^{-\frac{6s}{5\sqrt{2x}}\rho} e^{(8s-156)\frac{x}{\sqrt{2}}\rho} E_k P_k\left(\underline{\lambda}^1,\phi\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{5} V_n^{\frac{3}{5}s} g_d^{-2s} E_k P_k \left(\underline{\lambda}_4, \underline{\phi}\right) + \sum_{k=6}^{15} V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(2s-5)}{2}} V_n^{-\frac{3}{20}(2s-5)} g_d^{-\left(s-\frac{5}{2}\right)} V_n^{-\frac{1}{2}\left(s-\frac{5}{2}\right)} E_k P_k \left(\underline{\lambda}^2, \underline{\phi}\right) \\
&\quad + \sum_{k=16}^{25} V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(2s-5)}{2}} V_n^{-\frac{3}{20}(2s-5)} g_d^{\left(s-\frac{25}{2}\right)} V_n^{\frac{1}{2}\left(s-\frac{25}{2}\right)} E_k P_k \left(\underline{\lambda}_2, \underline{\phi}\right) \\
&\quad + V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(4s-30)}{2}} V_n^{-\frac{3}{20}(4s-30)} g_d^{-(2s-25)} V_n^{-\frac{1}{2}(2s-25)} E_{26} \\
&\quad + \sum_{k=27}^{36} V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(4s-30)}{2}} V_n^{-\frac{3}{20}(4s-30)} g_d^{-1} V_n^{-\frac{1}{2}} E_k P_k \left(\left(\underline{\lambda}^1 + \underline{\lambda}^4\right)^+, \underline{\phi}\right) \\
&\quad + V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(4s-30)}{2}} V_n^{-\frac{3}{20}(4s-30)} g_d^{-1} V_n^{-\frac{1}{2}} E_{37} + V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(4s-30)}{2}} V_n^{-\frac{3}{20}(4s-30)} g_d^{-1} V_n^{-\frac{1}{2}} E_{38} \\
&\quad + V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(4s-30)}{2}} V_n^{-\frac{3}{20}(4s-30)} g_d^{-1} V_n^{-\frac{1}{2}} E_{39} + V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(4s-30)}{2}} V_n^{-\frac{3}{20}(4s-30)} g_d^{-1} V_n^{-\frac{1}{2}} E_{40} \\
&\quad + V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(4s-30)}{2}} V_n^{-\frac{3}{20}(4s-30)} g_d^{-1} V_n^{-\frac{1}{2}} E_{41} + V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(4s-30)}{2}} V_n^{-\frac{3}{20}(4s-30)} g_d^{-1} V_n^{-\frac{1}{2}} E_{42} \\
&\quad + \sum_{k=43}^{52} V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(4s-30)}{2}} V_n^{-\frac{3}{20}(4s-30)} g_d^{-1} V_n^{-\frac{1}{2}} E_k P_k \left(\left(\underline{\lambda}^1 + \underline{\lambda}^4\right)^-, \underline{\phi}\right) \\
&\quad + V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(4s-30)}{2}} V_n^{-\frac{3}{20}(4s-30)} g_d^{(2s-53)} V_n^{\frac{1}{2}(2s-53)} E_{53} \\
&\quad + \sum_{k=54}^{63} V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(6s-83)}{2}} V_n^{-\frac{3}{20}(6s-83)} g_d^{-\left(s-\frac{53}{2}\right)} V_n^{-\frac{1}{2}\left(s-\frac{53}{2}\right)} E_k P_k \left(\underline{\lambda}^3, \underline{\phi}\right) \\
&\quad + \sum_{k=64}^{73} V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(6s-83)}{2}} V_n^{-\frac{3}{20}(6s-83)} g_d^{\left(s-\frac{73}{2}\right)} V_n^{\frac{1}{2}\left(s-\frac{73}{2}\right)} E_k P_k \left(\underline{\lambda}^3, \underline{\phi}\right) \\
&\quad + \sum_{k=74}^{78} V_n^{\frac{3}{5}s} g_d^{-2s} g_d^{\frac{(8s-156)}{2}} V_n^{-\frac{3}{20}(8s-156)} E_k P_k \left(\underline{\lambda}^3, \underline{\phi}\right) \\
&= \sum_{k=1}^{5} V_n^{\frac{3}{5}s} g_d^{-2s} E_k P_k \left(\underline{\lambda}_4, \underline{\phi}\right) + \sum_{k=6}^{15} V_n^{-\frac{1}{5}s+2} g_d^{-2s} E_k P_k \left(\underline{\lambda}^2, \underline{\phi}\right) \\
&\quad + \sum_{k=16}^{25} V_n^{\frac{4}{5}s-\frac{11}{2}} g_d^{-15} E_k P_k \left(\underline{\lambda}_2, \underline{\phi}\right) + V_n^{-s+17} g_d^{-2s+10} E_{26} \\
&\quad + \sum_{k=27}^{36} V_n^{4} g_d^{-16} E_k P_k \left(\left(\underline{\lambda}^1 + \underline{\lambda}^4\right)^+, \underline{\phi}\right) \\
&\quad + V_n^4 g_d^{-16} E_{37} + V_n^4 g_d^{-16} E_{38} + V_n^4 g_d^{-16} E_{39} + V_n^4 g_d^{-16} E_{40} \\
&\quad + V_n^4 g_d^{-16} E_{41} + V_n^4 g_d^{-16} E_{42} + \sum_{k=43}^{52} V_n^4 g_d^{-16} E_k P_k \left(\left(\underline{\lambda}^1 + \underline{\lambda}^4\right)^-, \underline{\phi}\right) \\
&\quad + V_n^{s+31} g_d^{2s-68} E_{53} + \sum_{k=54}^{63} V_n^{-\frac{4}{5}s+\frac{257}{10}} g_d^{-15} E_k P_k \left(\underline{\lambda}^3, \underline{\phi}\right) \\
&\quad + \sum_{k=64}^{73} V_n^{\frac{1}{5}s-\frac{29}{5}} g_d^{2s-78} E_k P_k \left(\underline{\lambda}^3, \underline{\phi}\right) \\
&\quad + \sum_{k=74}^{78} V_n^{-\frac{3}{5}s+\frac{117}{5}} g_d^{2s-78} E_k P_k \left(\underline{\lambda}^3, \underline{\phi}\right) .
\end{aligned}
\tag{C.5}
$$

$(j_n,\, j_{n+1})$	(0,0)	(1,1)	(1,2)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	(3,1)	(3,2)	(4,0)	
$SL(m)$ rep.	0	$\underline{\lambda}^2$	$\underline{\lambda}^2$	$\underline{\lambda}_1 + \underline{\lambda}_3$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{\lambda}_2$	$\underline{\lambda}_2$	0	0	$\underline{\lambda}^2$	$\underline{\lambda}^2$	$\underline{0}$	
$SL(2)$ weight	0	μ	$-\mu$	0	2μ	0	0	-2μ	μ	$-\mu$	0	-2μ	μ	$-\mu$	0	
d_α	1	6	6	1	6	1	1	1	1	1	6	1	6	6	1	
a_α	0	1	7	13	14	20	21	22	23	24	25	31	32	38	44	45
b_α	0	0	6	12	42	44	46	48	50	68	86	48	50	68	86	90
$n_c a_\alpha - b_\alpha$	0	1	1	14	14	14	14	14	14	14	14	14	46	46	90	

Table 15: Decomposition of the **45** of $SO(5,5)$

C.5 45 of $SO(5,5)$

The $SL(2) \times SL(n)$ decomposition of the representation of $SO(5,5)$ with highest weight $\vec{\Lambda}^n$ is given in table 15. The perturbative part of the unconstrained Eisenstein-like automorphic form constructed from the representation of $SO(5,5)$ with highest weight $\vec{\Lambda}^n$ is given by

$$\begin{aligned}
\Phi_p =& e^{-\frac{2s}{\sqrt{2}x}\rho} E_1 + \sum_{k=2}^{7} e^{-\frac{2s}{\sqrt{2}x}\rho} e^{(2s-1)\frac{x}{\sqrt{2}}\rho} e^{-\phi\left(s-\frac{1}{2}\right)} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ \sum_{k=8}^{13} e^{-\frac{2s}{\sqrt{2}x}\rho} e^{(2s-1)\frac{x}{\sqrt{2}}\rho} e^{\phi\left(s-\frac{13}{2}\right)} E_k P_k\left(\underline{\lambda}^2, \phi\right) + e^{-\frac{2s}{\sqrt{2}x}\rho} e^{(4s-14)\frac{x}{\sqrt{2}}\rho} e^{-\phi(2s-13)} E_{14} \\
&+ \sum_{k=15}^{20} e^{-\frac{2s}{\sqrt{2}x}\rho} e^{(4s-14)\frac{x}{\sqrt{2}}\rho} e^{-\phi} E_k P_k\left(\left(\underline{\lambda}^1 + \underline{\lambda}^3\right)^+, \underline{\phi}\right) \\
&+ \sum_{k=21}^{25} e^{-\frac{2s}{\sqrt{2}x}\rho} e^{(4s-14)\frac{x}{\sqrt{2}}\rho} e^{-\phi} E_k \\
&+ \sum_{k=26}^{31} e^{-\frac{2s}{\sqrt{2}x}\rho} e^{(4s-14)\frac{x}{\sqrt{2}}\rho} e^{-\phi} E_k P_k\left(\left(\underline{\lambda}^1 + \underline{\lambda}^3\right)^-, \underline{\phi}\right) \\
&+ e^{-\frac{2s}{\sqrt{2}x}\rho} e^{(4s-14)\frac{x}{\sqrt{2}}\rho} e^{\phi(2s-32)} E_{32} \\
&+ \sum_{k=33}^{38} e^{-\frac{2s}{\sqrt{2}x}\rho} e^{(6s-46)\frac{x}{\sqrt{2}}\rho} e^{-\phi(s-16)} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ \sum_{k=39}^{44} e^{-\frac{2s}{\sqrt{2}x}\rho} e^{(6s-46)\frac{x}{\sqrt{2}}\rho} e^{\phi(s-22)} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ e^{-\frac{2s}{\sqrt{2}x}\rho} e^{(8s-90)\frac{x}{\sqrt{2}}\rho} E_{45}
\end{aligned}$$

$$
\begin{aligned}
=&V_n^s g_d^{-2s} E_1 + \sum_{k=2}^{7} V_n^{-\frac{s}{2}} g_d^{-2s} g_d^{\frac{(2s-1)}{2}} V_n^{-\frac{1}{4}(2s-1)} g_d^{-\left(s-\frac{1}{2}\right)} V_n^{-\frac{1}{2}\left(s-\frac{1}{2}\right)} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ \sum_{k=8}^{13} V_n^s g_d^{-2s} g_d^{\frac{(2s-1)}{2}} V_n^{-\frac{1}{4}(2s-1)} g_d^{\left(s-\frac{13}{2}\right)} V_n^{\frac{1}{2}\left(s-\frac{13}{2}\right)} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) \\
&+ \sum_{k=15}^{20} V_n^s g_d^{-2s} g_d^{\frac{(4s-14)}{2}} V_n^{-\frac{1}{4}(4s-14)} g_d^{-1} V_n^{-\frac{1}{2}} E_k P_k\left(\left(\underline{\lambda}^1 + \underline{\lambda}^3\right)^+, \underline{\phi}\right) \\
&+ \sum_{k=21}^{25} V_n^s g_d^{-2s} g_d^{\frac{(4s-14)}{2}} V_n^{-\frac{1}{4}(4s-14)} g_d^{-1} V_n^{-\frac{1}{2}} E_k \\
&+ \sum_{k=26}^{31} V_n^s g_d^{-2s} g_d^{\frac{(4s-14)}{2}} V_n^{-\frac{1}{4}(4s-14)} g_d^{-1} V_n^{-\frac{1}{2}} E_k P_k\left(\left(\underline{\lambda}^1 + \underline{\lambda}^3\right)^-, \underline{\phi}\right) \\
&+ V_n^s g_d^{-2s} g_d^{\frac{(4s-14)}{2}} V_n^{-\frac{1}{4}(4s-14)} g_d^{2s-32} V_n^{\frac{(2s-32)}{2}} E_{32} \\
&+ \sum_{k=33}^{38} V_n^s g_d^{-2s} g_d^{\frac{(6s-46)}{2}} V_n^{-\frac{1}{4}(6s-46)} g_d^{-(s-16)} V_n^{-\frac{1}{2}(s-16)} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ \sum_{k=39}^{44} V_n^s g_d^{-2s} g_d^{\frac{(6s-46)}{2}} V_n^{-\frac{1}{4}(6s-46)} g_d^{(s-22)} V_n^{\frac{1}{2}(s-22)} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ V_n^s g_d^{-2s} g_d^{\frac{(8s-90)}{2}} V_n^{-\frac{1}{4}(8s-90)} E_{45} \\
=&V_n^s g_d^{-2s} E_1 + \sum_{k=2}^{7} V_n^{\frac{1}{2}} g_d^{-2s} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ \sum_{k=8}^{13} V_n^{s-3} g_d^{-7} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) \\
&+ V_n^{s-3} g_d^{-2s+6} E_{14} + \sum_{k=15}^{20} V_n^{-\frac{1}{2}s+\frac{7}{2}} g_d^{s-7} E_k P_k\left(\underline{\lambda}^1 + \underline{\lambda}^3, \underline{\phi}\right) \\
&+ \sum_{k=21}^{25} V_n^3 g_d^{-8} E_k \\
&+ \sum_{k=26}^{31} V_n^3 g_d^{-8} E_k P_k\left(\left(\underline{\lambda}^1 + \underline{\lambda}^3\right)^-, \underline{\phi}\right) \\
&+ V_n^{s+\frac{39}{2}} g_d^{2s-39} E_{32} \\
&+ \sum_{k=33}^{38} V_n^{-s+\frac{39}{2}} g_d^{-7} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ \sum_{k=39}^{44} V_n^{\frac{1}{2}} g_d^{2s-10} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ V_n^{-s+\frac{45}{2}} g_d^{2s-45} E_{45}.
\end{aligned} \tag{C.6}
$$

C.6 **248** of E_8

The $SL(2) \times SL(n)$ decomposition of the representation of E_8 with highest weight $\vec{\Lambda}^1$ is given in table 16. The perturbative part of the unconstrained Eisenstein-like automorphic form constructed

$(j_n,\, j_{n+1})$	(0,0)	(1,1)	(1,2)	(2,0)	(3,1)	(3,2)	(4,1)	(4,2)	(4,3)
$SL(m)$ rep.	$\underline{\lambda}^1$	$\underline{\lambda}^6$	$\underline{\lambda}^6$	$\underline{\lambda}_4$	$\underline{\lambda}_2$	$\underline{\lambda}_2$	$\underline{0}$	$\left(\underline{\lambda}^1+\underline{\lambda}^6\right)^+$	$\underline{0}$
$SL(2)$ weight	0	μ	$-\mu$	0	μ	$-\mu$	2μ	0	0
d_α	7	7	7	35	21	21	1	21	8
a_α	0	7	14	21	56	77	98	99	120
b_α	0	0	7	14	84	147	210	214	298
$n_c a_\alpha - b_\alpha$	0	7	7	28	84	84	182	182	182

$(j_n,\, j_{n+1})$	(4,4)	(4,5)	(5,1)	(5,2)	(6,0)	(7,1)	(7,2)	(8,0)	
$SL(m)$ rep.	$\left(\underline{\lambda}^1+\underline{\lambda}^6\right)^-$	$\underline{0}$	0	$\underline{\lambda}^5$	$\underline{\lambda}^3$	$\underline{\lambda}^1$	$\underline{\lambda}^1$	$\underline{\lambda}^6$	
$SL(2)$ weight	0	-2μ	μ	$-\mu$	0	μ	$-\mu$	0	
d_α	21	1	21	21	35	7	7	7	
a_α	128	149	150	171	192	227	234	241	248
b_α	330	414	418	523	628	838	887	936	992
$n_c a_\alpha - b_\alpha$	182	182	332	332	524	751	751	992	

Table 16: Decomposition of the **248** of E_8

from the representation of E_8 with highest weight $\vec{\Lambda}^1$ is given by

$$\begin{aligned}
\Phi_p = &\sum_{k=1}^{7} e^{-\frac{4s}{7\sqrt{2}x}\rho} E_k P_k\left(\underline{\lambda}^1,\underline{\phi}\right) + \sum_{k=8}^{14} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(2s-7)\frac{x}{\sqrt{2}}\rho} e^{-\phi\left(s-\frac{7}{2}\right)} E_k P_k\left(\underline{\lambda}^6,\underline{\phi}\right)\\
&+ \sum_{k=15}^{21} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(2s-7)\frac{x}{\sqrt{2}}\rho} e^{\phi\left(s-\frac{21}{2}\right)} E_k P_k\left(\underline{\lambda}^6,\underline{\phi}\right) + \sum_{k=22}^{56} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(4s-28)\frac{x}{\sqrt{2}}\rho} E_k P_k\left(\underline{\lambda}^4,\underline{\phi}\right)\\
&+ \sum_{k=57}^{77} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(6s-84)\frac{x}{\sqrt{2}}\rho} e^{-\phi(s-28)} E_k P_k\left(\underline{\lambda}^2,\underline{\phi}\right)\\
&+ \sum_{k=78}^{98} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(6s-84)\frac{x}{\sqrt{2}}\rho} e^{\phi(s-49)} E_k P_k\left(\underline{\lambda}^2,\underline{\phi}\right) + e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(8s-182)\frac{x}{\sqrt{2}}\rho} e^{-\phi(2s-98)} E_{99}\\
&+ \sum_{k=100}^{120} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(8s-182)\frac{x}{\sqrt{2}}\rho} e^{-\phi} E_k P_k\left(\left(\underline{\lambda}^1+\underline{\lambda}^6\right)^+,\underline{\phi}\right)\\
&+ \sum_{k=121}^{128} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(8s-182)\frac{x}{\sqrt{2}}\rho} e^{-\phi} E_k\\
&+ \sum_{k=129}^{149} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(8s-182)\frac{x}{\sqrt{2}}\rho} e^{-\phi} E_k P_k\left(\left(\underline{\lambda}^1+\underline{\lambda}^6\right)^-,\underline{\phi}\right)\\
&+ e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(8s-182)\frac{x}{\sqrt{2}}\rho} e^{\phi(2s-150)} E_{150} + \sum_{k=151}^{171} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(10s-332)\frac{x}{\sqrt{2}}\rho} e^{-\phi(s-75)} E_k P_k\left(\underline{\lambda}^5,\underline{\phi}\right)\\
&+ \sum_{k=172}^{192} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(10s-332)\frac{x}{\sqrt{2}}\rho} e^{\phi(s-96)} E_k P_k\left(\underline{\lambda}^5,\underline{\phi}\right)\\
&+ \sum_{k=193}^{227} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(12s-524)\frac{x}{\sqrt{2}}\rho} E_k P_k\left(\underline{\lambda}^3,\underline{\phi}\right)\\
&+ \sum_{k=228}^{234} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(14s-751)\frac{x}{\sqrt{2}}\rho} e^{-\phi\left(s-\frac{227}{2}\right)} E_k P_k\left(\underline{\lambda}^1,\underline{\phi}\right)\\
&+ \sum_{k=235}^{241} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(14s-751)\frac{x}{\sqrt{2}}\rho} e^{\phi\left(s-\frac{241}{2}\right)} E_k P_k\left(\underline{\lambda}^1,\underline{\phi}\right) + \sum_{k=242}^{248} e^{-\frac{4s}{7\sqrt{2}x}\rho} e^{(14s-992)\frac{x}{\sqrt{2}}\rho} E_k P_k\left(\underline{\lambda}^6,\underline{\phi}\right)
\end{aligned} \tag{C.7}$$

$$
\begin{aligned}
\Phi_p =& \sum_{k=1}^{7} V_n^{\frac{2}{7}s} g_d^{-4s} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=8}^{14} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(2s-7)}{2}} V_n^{-\frac{1}{28}(2s-7)} g_d^{-\left(s-\frac{7}{2}\right)} V_n^{-\frac{1}{2}\left(s-\frac{7}{2}\right)} E_k P_k\left(\underline{\lambda}^6, \underline{\phi}\right) \\
&+ \sum_{k=15}^{21} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(2s-7)}{2}} V_n^{-\frac{1}{28}(2s-7)} g_d^{\left(s-\frac{21}{2}\right)} V_n^{\frac{1}{2}\left(s-\frac{21}{2}\right)} E_k P_k\left(\underline{\lambda}^6, \underline{\phi}\right) \\
&+ \sum_{k=22}^{56} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(4s-28)}{2}} V_n^{-\frac{1}{28}(4s-28)} E_k P_k\left(\underline{\lambda}_4, \underline{\phi}\right) \\
&+ \sum_{k=57}^{77} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(6s-84)}{2}} V_n^{-\frac{1}{28}(6s-84)} g_d^{-(s-28)} V_n^{-\frac{1}{2}(s-28)} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ \sum_{k=78}^{98} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(6s-84)}{2}} V_n^{-\frac{1}{28}(6s-84)} g_d^{(s-49)} V_n^{-\frac{1}{2}(s-49)} E_k P_k\left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(8s-182)}{2}} V_n^{-\frac{1}{28}(8s-182)} g_d^{-(2s-98)} V_n^{-\frac{1}{2}(2s-98)} E_{99} \\
&+ \sum_{k=100}^{120} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(8s-182)}{2}} V_n^{-\frac{1}{28}(8s-182)} g_d^{-1} V_n^{-\frac{1}{2}} E_k P_k\left(\left(\underline{\lambda}^1 + \underline{\lambda}^6\right)^+, \underline{\phi}\right) \\
&+ \sum_{k=100}^{120} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(8s-182)}{2}} V_n^{-\frac{1}{28}(8s-182)} g_d^{-1} V_n^{-\frac{1}{2}} E_k P_k\left(\left(\underline{\lambda}^1 + \underline{\lambda}^6\right)^+, \underline{\phi}\right) \\
&+ \sum_{k=121}^{128} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(8s-182)}{2}} V_n^{-\frac{1}{28}(8s-182)} g_d^{-1} V_n^{-\frac{1}{2}} E_k \\
&+ \sum_{k=129}^{149} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(8s-182)}{2}} V_n^{-\frac{1}{28}(8s-182)} g_d^{-1} V_n^{-\frac{1}{2}} E_k P_k\left(\left(\underline{\lambda}^1 + \underline{\lambda}^6\right)^-, \underline{\phi}\right) \\
&+ V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(8s-182)}{2}} V_n^{-\frac{1}{28}(8s-182)} g_d^{(2s-150)} V_n^{\frac{1}{2}(2s-150)} E_{150} \\
&+ \sum_{k=151}^{171} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(10s-332)}{2}} V_n^{-\frac{1}{28}(10s-332)} g_d^{-(s-75)} V_n^{-\frac{1}{2}(s-75)} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right) \\
&+ \sum_{k=172}^{192} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(10s-332)}{2}} V_n^{-\frac{1}{28}(10s-332)} g_d^{(s-96)} V_n^{\frac{1}{2}(s-96)} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right) \\
&+ \sum_{k=193}^{227} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(12s-524)}{2}} V_n^{-\frac{1}{28}(12s-524)} E_k P_k\left(\underline{\lambda}^3, \underline{\phi}\right) \\
&+ \sum_{k=228}^{234} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(14s-751)}{2}} V_n^{-\frac{1}{28}(14s-751)} g_d^{-\left(s-\frac{227}{2}\right)} V_n^{-\frac{1}{2}\left(s-\frac{227}{2}\right)} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) \\
&+ \sum_{k=235}^{241} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(14s-751)}{2}} V_n^{-\frac{1}{28}(14s-751)} g_d^{\left(s-\frac{241}{2}\right)} V_n^{\frac{1}{2}\left(s-\frac{241}{2}\right)} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) \\
&+ \sum_{k=242}^{248} V_n^{\frac{2}{7}s} g_d^{-4s} g_d^{\frac{(16s-992)}{2}} V_n^{-\frac{1}{28}(16s-992)} E_k P_k\left(\underline{\lambda}^6, \underline{\phi}\right)
\end{aligned} \tag{C.8}
$$

$$
\begin{aligned}
\Phi_p =& \sum_{k=1}^{7} V_n^{\frac{2s}{7}} g_d^{-4s} E_k P_k \left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=8}^{14} V_n^{-\frac{2}{7}s+2} g_d^{-4s} E_k P_k \left(\underline{\lambda}^6, \underline{\phi}\right) \\
&+ \sum_{k=15}^{21} V_n^{\frac{5}{7}s-5} g_d^{-2s-14} E_k P_k \left(\underline{\lambda}^6, \underline{\phi}\right) \\
&+ \sum_{k=22}^{56} V_n^{\frac{1}{7}s+1} g_d^{-2s-14} E_k P_k \left(\underline{\lambda}^4, \underline{\phi}\right) \\
&+ \sum_{k=57}^{77} V_n^{-\frac{3}{7}s+17} g_d^{-2s-14} E_k P_k \left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ \sum_{k=78}^{98} V_n^{\frac{4}{7}s+\frac{43}{2}} g_d^{-91} E_k P_k \left(\underline{\lambda}^2, \underline{\phi}\right) \\
&+ V_n^{-s+\frac{111}{2}} g_d^{-2s+7} E_{99} \\
&+ \sum_{k=100}^{120} V_n^6 g_d^{-92} E_k P_k \left(\left(\underline{\lambda}^1 + \underline{\lambda}^6\right)^+, \underline{\phi}\right) \\
&+ \sum_{k=121}^{128} V_n^6 g_d^{-92} E_k \\
&+ \sum_{k=129}^{149} V_n^6 g_d^{-92} E_k P_k \left(\left(\underline{\lambda}^1 + \underline{\lambda}^6\right)^-, \underline{\phi}\right) \\
&+ V_n^{s-\frac{137}{2}} g_d^{2s-241} E_{150} \\
&+ \sum_{k=151}^{171} V_n^{-\frac{2}{7}s+\frac{691}{14}} g_d^{-91} E_k P_k \left(\underline{\lambda}^5, \underline{\phi}\right) \\
&+ \sum_{k=172}^{192} V_n^{\frac{3}{7}s-\frac{253}{7}} g_d^{2s-262} E_k P_k \left(\underline{\lambda}^5, \underline{\phi}\right) \\
&+ \sum_{k=193}^{227} V_n^{-\frac{1}{7}s+\frac{131}{7}} g_d^{2s-262} E_k P_k \left(\underline{\lambda}^3, \underline{\phi}\right) \\
&+ \sum_{k=228}^{234} V_n^{-\frac{5}{7}s+\frac{585}{7}} g_d^{2s-262} E_k P_k \left(\underline{\lambda}^1, \underline{\phi}\right) \\
&+ \sum_{k=235}^{241} V_n^{\frac{2}{7}s-\frac{234}{7}} g_d^{4s-496} E_k P_k \left(\underline{\lambda}^1, \underline{\phi}\right) \\
&+ \sum_{k=242}^{248} V_n^{-\frac{2}{7}s+\frac{248}{7}} g_d^{4s-496} E_k P_k \left(\underline{\lambda}^6, \underline{\phi}\right) .
\end{aligned} \tag{C.9}
$$

C.7 56 of E_7

The $SL(2) \times SL(n)$ decomposition of the representation of E_7 with highest weight $\vec{\Lambda}^1$ is given in table 17. The perturbative part of the unconstrained Eisenstein-like automorphic form constructed

$(j_n,\, j_{n+1})$	(0,0)	(1,1)	(1,2)	(2,0)	(3,1)	(3,2)	(4,0)	
$SL(m)$ rep. $SL(2)$ weight	$\underline{\lambda}^1$ 0	$\underline{\lambda}^5$ μ	$\underline{\lambda}^5$ $-\mu$	$\underline{\lambda}^3$ 0	$\underline{\lambda}^1$ μ	$\underline{\lambda}^1$ $-\mu$	$\underline{\lambda}^5$ 0	
d_α	6	6	6	20	6	6	6	
a_α	0	6	12	18	38	44	50	56
b_α	0	0	6	12	52	70	88	112
$n_c a_\alpha - b_\alpha$	0	6	6	24	62	62	112	

Table 17: Decomposition of the **56** of E_7

from the representation of E_7 with highest weight $\vec{\Lambda}^1$ is given by

$$\begin{aligned}
\Phi_p =& \sum_{k=1}^{6} e^{-\frac{2s}{3\sqrt{2x}}\rho} E_k P_k\left(\underline{\lambda}^1, \phi\right) + \sum_{k=12}^{7} e^{-\frac{2s}{3\sqrt{2x}}\rho} e^{(2s-6)\frac{x}{\sqrt{2}}\rho} e^{-\phi(s-3)} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right) \\
&+ \sum_{k=13}^{18} e^{-\frac{2s}{3\sqrt{2x}}\rho} e^{(2s-6)\frac{x}{\sqrt{2}}\rho} e^{\phi(s-9)} E_k P_k\left(\underline{\lambda}^5, \phi\right) + \sum_{k=19}^{38} e^{-2\frac{s}{3\sqrt{2x}}\rho} e^{(4s-24)\frac{x}{\sqrt{2}}\rho} E_k P_k\left(\underline{\lambda}^3, \underline{\phi}\right) \\
&+ \sum_{k=39}^{44} e^{-\frac{2s}{3\sqrt{2x}}\rho} e^{(6s-62)\frac{x}{\sqrt{2}}\rho} e^{-\phi(s-19)} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) \\
&+ \sum_{k=45}^{50} e^{-\frac{2s}{3\sqrt{2x}}\rho} e^{(6s-62)\frac{x}{\sqrt{2}}\rho} e^{\phi(s-25)} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) \\
&+ \sum_{k=51}^{56} e^{-\frac{2s}{3\sqrt{2x}}\rho} e^{(8s-112)\frac{x}{\sqrt{2}}\rho} E_k P_k\left(\underline{\lambda}^5, \phi\right) \\
=& \sum_{k=1}^{6} V_n^{\frac{s}{3}} g_d^{-2s} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=7}^{12} V_n^{\frac{s}{3}} g_d^{-2s} g_d^{\frac{(2s-6)}{2}} V_n^{-\frac{1}{12}(2s-6)} g_d^{-(s-3)} V_n^{-\frac{1}{2}(s-3)} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right) \\
&+ \sum_{k=13}^{18} V_n^{\frac{s}{3}} g_d^{-2s} g_d^{\frac{(2s-6)}{2}} V_n^{-\frac{1}{12}(2s-6)} g_d^{(s-9)} V_n^{\frac{1}{2}(s-9)} E_k P_k\left(\underline{\lambda}_5, \underline{\phi}\right) \\
&+ \sum_{k=19}^{38} V_n^{\frac{s}{3}} g_d^{-2s} g_d^{\frac{(4s-24)}{2}} V_n^{-\frac{1}{12}(4s-24)} E_k P_k\left(\underline{\lambda}^3, \underline{\phi}\right) \\
&+ \sum_{k=39}^{44} V_n^{\frac{s}{3}} g_d^{-2s} g_d^{\frac{(6s-62)}{2}} V_n^{-\frac{1}{12}(6s-62)} g_d^{-(s-19)} V_n^{-\frac{1}{2}(s-19)} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) \\
&+ \sum_{k=45}^{50} V_n^{\frac{s}{3}} g_d^{-2s} g_d^{\frac{(6s-62)}{2}} V_n^{-\frac{1}{12}(6s-62)} g_d^{(s-25)} V_n^{\frac{1}{2}(s-25)} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) \\
&+ \sum_{k=51}^{56} V_n^{\frac{s}{3}} g_d^{-2s} g_d^{\frac{(8s-112)}{2}} V_n^{-\frac{1}{12}(8s-112)} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right) \\
=& \sum_{k=1}^{6} V_n^{\frac{s}{3}} g_d^{-2s} E_k P_k\left(\underline{\lambda}_1, \underline{\phi}\right) + \sum_{k=7}^{12} V_n^{-\frac{1}{3}s+2} g_d^{-2s} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right) \\
&+ \sum_{k=13}^{18} V_n^{\frac{2}{3}s-4} g_d^{-12} E_k P_k\left(\underline{\lambda}_2, \underline{\phi}\right) + \sum_{k=19}^{38} V_n^{2} g_d^{-12} E_k P_k\left(\underline{\lambda}^3, \underline{\phi}\right) \\
&+ \sum_{k=39}^{44} V_n^{-\frac{2}{3}s+\frac{44}{3}} g_d^{-12} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) + \sum_{k=45}^{50} V_n^{\frac{1}{3}s-\frac{22}{3}} g_d^{2s-56} E_k P_k\left(\underline{\lambda}^1, \underline{\phi}\right) \\
&+ \sum_{k=51}^{56} V_n^{-\frac{1}{3}s+\frac{28}{3}} g_d^{2s-56} E_k P_k\left(\underline{\lambda}^5, \underline{\phi}\right).
\end{aligned} \tag{C.10}$$

D Decomposition of the E_{n+1} Algebra in Various Limits

D.1 Perturbative Limit

To investigate the properties of our automorphic form in the $g_d \to 0$ limit it will be expedient to decompose the E_{n+1} algebra into a $GL(1) \times SO(n,n)$ subalgebra. To do this we delete node $n+1$ of the Dynkin diagram, the simple roots $\vec{\alpha}$ of E_{n+1} then decompose as

$$\begin{aligned} \vec{\alpha}_i &= (0, \tilde{\alpha}_i)\,, \quad i = 1, ..., n \\ \vec{\alpha}_{n+1} &= \left(x, -\tilde{\lambda}_n\right), \end{aligned} \tag{D.1}$$

where the tilde denotes $SO(n)$ simple roots and fundamental weights. The variable x is fixed by the condition on the length of the simple roots, $\vec{\alpha}_{n+1}^2 = 2 = x^2 + \lambda_n^2$, this leads to $x^2 = \frac{8-n}{4}$. The corresponding fundamental weights are

$$\begin{aligned} \vec{\Lambda}^i &= \left(\frac{\tilde{\lambda}_i . \tilde{\lambda}_n}{x}, \tilde{\lambda}_i\right), \quad i = 1, ..., n, \\ \vec{\Lambda}^{n+1} &= \left(\frac{1}{x}, \tilde{0}\right). \end{aligned} \tag{D.2}$$

D.2 M-theory Limit

To investigate the properties of our automorphic form in the $V_m \to \infty$ limit we decompose the E_{n+1} algebra into a $GL(1) \times SL(n+1)$ subalgebra. To do this we delete node $n-1$ of the Dynkin diagram, the simple roots $\vec{\alpha}$ of E_{n+1} then decompose as

$$\begin{aligned} \vec{\alpha}_i &= (0, \underline{\alpha}_i)\,, \quad i = 1, ..., n-2, \\ \vec{\alpha}_{n-1} &= \left(x, -\underline{\lambda}_{n-2}\right) \\ \vec{\alpha}_n &= \left(0, \underline{\alpha}_{n-1}\right) \\ \vec{\alpha}_{n+1} &= (0, \underline{\alpha}_n) \end{aligned} \tag{D.3}$$

where the underline denotes $SL(n+1)$ simple roots and fundamental weights. The variable x is fixed by the condition on the length of the simple roots, $\vec{\alpha}_{n-1}^2 = 2 = x^2 + \underline{\lambda}_{n-2}^2$, this leads to

$x^2 = \frac{8-n}{n+1}$. The corresponding fundamental weights are

$$\begin{aligned}
\vec{\Lambda}^i &= \left(\frac{\underline{\lambda}_i \cdot \underline{\lambda}_{n-2}}{x}, \underline{\lambda}_i\right), \quad i = 1, ..., n, \\
\vec{\Lambda}^{n-1} &= \left(\frac{1}{x}, \underline{0}\right), \\
\vec{\Lambda}^n &= \left(\frac{\underline{\lambda}_{n-1} \cdot \underline{\lambda}_{n-2}}{x}, \underline{\lambda}_{n-1}\right), \\
\vec{\Lambda}^{n+1} &= \left(\frac{\underline{\lambda}_n \cdot \underline{\lambda}_{n-2}}{x}, \underline{\lambda}_n\right).
\end{aligned} \tag{D.4}$$

D.3 IIB Volume Limit

To investigate the properties of our automorphic form in the $V_m \to \infty$ limit we decompose the E_{n+1} algebra into a $GL(1) \times SL(2) \times SL(n)$ subalgebra. To do this we delete node n of the Dynkin diagram, the simple roots $\vec{\alpha}$ of E_{n+1} then decompose as

$$\begin{aligned}
\vec{\alpha}_i &= (0, 0, \underline{\alpha}_i), \quad i = 1, ..., n-1, \\
\vec{\alpha}_n &= \left(x, -\mu_1, -\underline{\lambda}_{n-2}\right), \\
\vec{\alpha}_{n+1} &= (0, \beta_1, \underline{0}),
\end{aligned} \tag{D.5}$$

where the underline denotes $SL(n)$ simple roots and fundamental weights and μ_1, β_1 are the fundamental weight and simple root of $SL(2)$ respectively. The variable x is fixed by the condition on the length of the simple roots, $\vec{\alpha}_n^2 = 2 = x^2 + \underline{\lambda}_{n-2}^2 + \mu_1^2$, this leads to $x^2 = \frac{8-n}{2n}$. The corresponding fundamental weights are

$$\begin{aligned}
\vec{\Lambda}^i &= \left(\frac{\underline{\lambda}_i \cdot \underline{\lambda}_{n-2}}{x}, 0, \underline{\lambda}_i\right), \quad i = 1, ..., n-1, \\
\vec{\Lambda}^n &= \left(\frac{1}{x}, 0, \underline{0}\right) \\
\vec{\Lambda}^{n+1} &= \left(\frac{1}{2x}, \mu_1, \underline{0}\right).
\end{aligned} \tag{D.6}$$

D.4 Decompactification of a Single Dimension Limit

To investigate the properties of our automorphic form in the $V_m \to \infty$ limit we decompose the E_{n+1} algebra into a $GL(1) \times E_n$ subalgebra. To do this we delete node 1 of the Dynkin diagram, the simple roots $\vec{\alpha}$ of E_{n+1} then decompose as

$$\begin{aligned}
\vec{\alpha}_1 &= \left(x, -\hat{\lambda}_1\right), \\
\vec{\alpha}_i &= (0, \hat{\alpha}_{i-1}), \quad i = 2, ..., n+1,
\end{aligned} \tag{D.7}$$

where the hat denotes E_n simple roots and fundamental weights. The variable x is fixed by the condition on the length of the simple roots, $\vec{\alpha}_1^2 = 2 = x^2 + \hat{\lambda}_1^2$. The corresponding fundamental weights are

$$\begin{aligned} \vec{\Lambda}^1 &= \left(\frac{1}{x}, \underline{0}\right), \\ \vec{\Lambda}^i &= \left(\frac{\hat{\lambda}_{i-1}.\hat{\lambda}_1}{x}, \hat{\lambda}_{i-1}\right), \quad i = 2, ..., n+1. \end{aligned} \tag{D.8}$$

We now proceed to calculate the inner products of the E_n fundamental weights. To do this we decompose the E_n algebra into a $GL(1) \times SL(n)$ subalgebra by deleting node $n-1$, one finds

$$\begin{aligned} \hat{\alpha}_i &= (0, -\underline{\alpha}_i), \quad i = 1, ..., n-3, \\ \hat{\alpha}_{n-2} &= \left(y, -\underline{\lambda}_{n-3}\right), \\ \hat{\alpha}_{n-1} &= \left(0, \underline{\alpha}_{n-2}\right), \\ \hat{\alpha}_{n-1} &= \left(0, \underline{\alpha}_{n-1}\right), \end{aligned} \tag{D.9}$$

with fundamental weights

$$\begin{aligned} \hat{\lambda}_i &= \left(\frac{\underline{\lambda}_i.\underline{\lambda}_{n-3}}{y}, \underline{\lambda}_i\right), \quad i = 1, ..., n-3, \\ \hat{\lambda}_{n-2} &= \left(\frac{1}{y}, \underline{0}\right), \\ \hat{\lambda}_{n-1} &= \left(\frac{\underline{\lambda}_{n-2}.\underline{\lambda}_{n-3}}{y}, \underline{\lambda}_{n-2}\right), \\ \hat{\lambda}_n &= \left(\frac{\underline{\lambda}_{n-1}.\underline{\lambda}_{n-3}}{y}, \underline{\lambda}_{n-1}\right). \end{aligned} \tag{D.10}$$

The variable y is fixed by the condition $\hat{\alpha}_{n-2}^2 = 2$, this gives $y^2 = \frac{9-n}{n}$. We then have

$$\begin{aligned} \hat{\lambda}_1.\hat{\lambda}_1 &= \left(\frac{3}{ny}, \underline{\lambda}_1\right) \cdot \left(\frac{3}{ny}, \underline{\lambda}_1\right) \\ &= \frac{9}{n^2 y^2} + \frac{n-1}{n} \\ &= \frac{10-n}{9-n}, \end{aligned} \tag{D.11}$$

where we have made use of the expression $\underline{\lambda}_i.\underline{\lambda}_j = \frac{i(n-j)}{n}$ for $i \le j$. We may now substitute this back into $\vec{\alpha}_1.\vec{\alpha}_1$ to fix the variable x,

$$\begin{aligned} x^2 &= 2 - \hat{\lambda}_1.\hat{\lambda}_1 \\ &= \frac{8-n}{9-n}. \end{aligned} \tag{D.12}$$

D.5 IIA Volume Limit

The decomposition of representations of E_{n+1} into those of $SL(n) \times GL(1) \times GL(1)$ is given by deleting nodes n and $n+1$ of the Dynkin diagram appropriate to the type IIA theory. In this section we will find how the roots and weights of E_{n+1} decompose in terms of those of $SL(n) \times GL(1) \times GL(1)$.

Let us carry out the decomposition by first deleting node n to find the roots and fundamental weights of D_n and then delete node $n+1$ to find the algebra $SL(n)$. Using the methods given in reference, the simple roots of E_{n+1} can be expressed as

$$\vec{\alpha}_i = (0, \underline{\tilde{\alpha}}_i)\,, \quad i = 1, ..., n-1, n+1, \quad \vec{\alpha}_n = \left(x, -\underline{\tilde{\lambda}}_{n-1}\right). \tag{D.13}$$

Here $\underline{\tilde{\alpha}}_i, i = 1, ..., n$ are the roots of D_n and $\underline{\tilde{\lambda}}_i$ are its fundamental weights which are given by

$$\vec{\Lambda}_i = \left(\frac{\underline{\tilde{\lambda}}_i \cdot \underline{\tilde{\lambda}}_{n-1}}{x}, \underline{\tilde{\lambda}}_i\right), \quad i = 1, ..., n-1, n+1, \quad \vec{\Lambda}_n = \left(\frac{1}{x}, \underline{0}\right). \tag{D.14}$$

The variable x is fixed by demanding that $\alpha_n^2 = 2 = x^2 + \underline{\tilde{\lambda}}_{n-1}^2$.

We now delete node n to find the A_{n-1} algebra. The roots of E_{n+1} are found from the above roots by substituting the corresponding decomposition of the D_n roots and weights into those of A_{n-1}. The roots of D_n in terms of those of A_{n-1} are given by $\underline{\tilde{\alpha}}_i = (0, \underline{\alpha}_i)\,,\ i = 1, ..., n-1$ and $\underline{\tilde{\alpha}}_n = \left(y, -\underline{\lambda}_{n-2}\right)$ while the fundamental weights are given by $\underline{\tilde{\lambda}}_i = \left(\frac{\lambda_{n-2} \cdot \lambda_i}{y}, \underline{\lambda}_i\right)$ $i = 1, ..., n-1$ and $\underline{\tilde{\lambda}}_{n+1} = \left(\frac{1}{y}, \underline{0}\right)$. Requiring $\tilde{\alpha}_{n+1}^2 = 2$ gives $y^2 = \frac{4}{n}$ We then find that the roots of E_{n+1} are given by

$$\begin{aligned} \vec{\alpha}_i &= (0, 0, \underline{\alpha}_i)\,, \quad i = 1, ..., n-1, \\ \vec{\alpha}_n &= \left(x, -\frac{\lambda_{n-2} \cdot \lambda_{n-1}}{y}, -\underline{\lambda}_{n-1}\right), \\ \vec{\alpha}_{n+1} &= \left(0, y, -\underline{\lambda}_{n-2}\right). \end{aligned} \tag{D.15}$$

The fundamental weights of E_{n+1} are found in the same way to be

$$\vec{\Lambda}_i = \left(\frac{c_i}{x}, \frac{\lambda_{n-2} \cdot \lambda_i}{y}, \underline{\lambda}_i\right), \quad i = 1, ..., n-1, \tag{D.16}$$

$$\vec{\Lambda}_n = \left(\frac{1}{x}, 0, \underline{0}\right), \tag{D.17}$$

$$\vec{\Lambda}_{n+1} = \left(\frac{n-2}{4x}, \frac{1}{y}, \underline{0}\right), \tag{D.18}$$

where $c_i = \frac{i}{2},\ i = 1, \ldots, n-2$ and $c_{n-1} = \frac{n}{4}$. As $\underline{\tilde{\lambda}}_{n-1}^2 = \frac{n}{4}$ we find that $x^2 = \frac{8-n}{4}$.

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